THE ELASTIC PROPERTIES OF RANDOM NETWORKS

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ABSTRACT

The elastic properties of random networks of springs is richer than the analogous problem with resistors. It is shown that there are two classes of problem. As springs are cut, the network either breaks up at \( p^* = p_c (\text{class 1}) \) or at \( p^* > p_c (\text{class 2}) \) where \( p \) is the fraction of springs present. Class 1 is ordinary geometrical percolation but class 2 is better viewed as rigidity percolation. An important example of the latter is provided by covalent random networks.

INTRODUCTION

The study of the properties of random elastic systems has been pursued for the past year by a number of groups. A clear picture has yet to emerge and in this brief summary I should like to raise some of the important issues and give a list of references.

It is surprising that this subject is only being pursued now. The random resistor problem has been extensively studied and the reader is referred to the review articles by Kirkpatrick [1] and Stailey [2]. In the simplest (class 1) version of this problem, a network of mass points is connected together and described by a potential that couples the (small) motions of the mass points. Sufficient forces must be specified so that when parts of the network are removed, geometrical connection implies elastic connection. In two dimensions this means specifying say, nearest neighbor central forces involving all bonds (ij) and also all angular forces involving the bonds (ij) and (jk). In three dimensions, the dihedral angle forces involving bonds (ij), (jk) and (kl) must be specified in addition. The elastic properties of these networks vanish at percolation \( p_c \), when the network becomes geometrically disconnected. This can be achieved by randomly removing a fraction \((1-p)\) of sites or bonds. It can also be accomplished by cutting holes in a sheet of material like aluminum [3] in the single real experiment that has been done to date.
In class 2 problems, fewer forces are specified and the transition takes place at \( p^* > p_c \) which corresponds to rigidity percolation. Much less is known about class 2 problems but there is an important microscopic realization in covalent network glasses. We will briefly discuss these topics in the next few sections.

Class 1 Systems

The underlying geometry is ordinary geometrical percolation but there are two new elastic quantities to monitor in isotropic systems. Simulations have been done in two dimensions on the triangular net [4] and on the honeycomb lattice [5] with bonds randomly removed. The results show that all the elastic constants go to zero at \( p_c \) with a universal exponent \( f \)

\[
C_{ij} \sim (p-p_c)^f
\]

with

\[ f = 3.5 \pm 0.2 \]  \hspace{1cm} (1)

The value of \( f \) in 3D is not known. We note that this exponent is quite different from \( t = 1.3 \) for the behavior of the conductivity near \( p_c \) in resistor networks [6]. The ratio of the two critical amplitudes \( \nu_s \) (longitudinal and velocity) and \( \nu_t \) (transverse sound velocity) approaches a universal value at \( p_c \) [5]

\[
\nu_s^2/\nu_t^2 = 3.5 \pm 0.2
\]  \hspace{1cm} (2)

It is not known whether the equality of (1) and (2) is a coincidence. It is remarkable that (1) is predicted to be universal, that is, independent of the geometry of the lattice and of the initial value of \( \nu_s^2/\nu_t^2 \) when all bonds are present. Continuum effective medium theories [7] show that if ellipses with an aspect ratio of \( r = b/a \) (the ratio of major axes) are randomly punched into a 2D continuum then

\[
\nu_s^2/\nu_t^2 = 2[1 + r/(r^2 + 1)]
\]  \hspace{1cm} (3)

as percolation is approached.

A major unresolved issue is whether square (or cubic) systems become isotropic as percolation is approached. This would imply that the three elastic constants of the square (or cubic) system satisfy the condition that the anisotropy parameter \( A \) tends to zero at \( p_c \), where

\[ A = (C_{11}-C_{12}-2C_{44})/C_{11} \]  \hspace{1cm} (4)

That is the leading order singularities, involving \( f \), cancel in the numerator. By analogy with magnetic systems, we might expect isotropy at \( p_c \), but this has not yet been shown.
Class 2 Systems

Much less is known about these systems. Examples are the triangular net and f.c.c. lattices with central forces only [8]. As bonds are removed, local regions become floppy even though they are geometrically connected [9]. The network loses its elastic properties at \( p^* > p_c \) so that it has no elastic properties even though remaining geometrically connected. Constraints arguments show that for central forces

\[
p^* = 2z/d
\]

(5)

Numerical simulations show that (5) is very accurate and could even be exact [10] for reasons that are unknown. Indeed effective medium theory agrees almost exactly with numerical simulations except for a small region around \( p^* \). It is not clear whether the small "tail" seen in the simulations is intrinsic or due to finite size effects. The ratio of elastic constants appear to be independent of \( p \) (as predicted by effective medium theory).

There is disagreement on the value of \( f \). This is closely tied in with the size of the tail and is hard to estimate from simulations. Feng and Sen [11] find \( f \approx 2.4 \pm 0.4 \) whereas Lemieux et al. [12] find that \( f \approx 1.4 \pm 0.2 \) (effective medium theory gives \( f = 1 \)). It is certain that class 2 problems are in a different universality class from class 1. The node link model of percolation [13] has been extended to describe class 1 problems and gives \( f = 11/3 = 3.67 \) in close agreement with (1). It is possible that this model could be extended to class 2 problems in order to clarify the underlying physics.

Covalent Glasses

An important application of class 2 problems is to covalent network glasses like Ge-Se, where \( 0 < x < 1 \). These glasses have covalent bonds (Ge has 4 bonds and Se has 2 bonds) and mean co-ordination \( < r > = 2x + 2 \). Constraints arguments [14,9] on networks with nearest neighbor central and angle forces (but no dihedral angle forces) suggest a phase transition at

\[
<r> = r_p = 2.4
\]

(6)

For \( < r > < r_p \), the network is floppy. As \( < r > \) increases the rigid regions join together and rigidity percolates at \( r_p \). It is, therefore, predicted that there are two kinds of covalent glasses. Those with \( < r > < r_p \) may be referred to as polymeric glasses as they can be thought of as chains with a few cross links. Those in the \( < r > > r_p \) may be called amorphous solids.

Recent simulations of the elastic properties [15] confirm these ideas and show that

\[
C_{ij} \sim (<r> - r_p)^f
\]

where

\[
r_p = 2.4
\]

(7)
and

\[ f = 1.5 \]

The experimental implications of these ideas are not yet fully developed. These glasses provide the only microscopic system in which elastic problems of either class 1 or class 2 can be studied.

CONCLUSIONS

As you can see from the number of preprints referenced, this is a rapidly developing area. Many of the issues raised here should become clearer within the next year or so.

REFERENCES