BETHE LATTICES

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1. INTRODUCTION

A great many problems in theoretical physics, involving interacting many particle systems, are at some stage "solved exactly on the Bethe lattice." While this is a delight to the theoretician because he knows exactly what has been done, it is usually a source of some annoyance to the experimentalist. This annoyance is usually increased when it is explained that this is not a real lattice but a pseudo-lattice.

In this lecture I hope to give you some idea of the many diverse areas in which the Bethe lattice is used and of the similarity in the theoretical techniques employed. Other speakers will be going into much more detail in specific cases. I have particularly enjoyed preparing these notes as I do not think all these applications of the Bethe lattice have been collected together in one place before.

In the next section the history of the Bethe lattice is traced. In section 3, the Bethe approximation in alloys and the Ising model is reviewed. In section 4, the percolation problem is solved on the Bethe lattice. In section 5, a tight binding Hamiltonian is solved and in section 6, the vibrational properties of the Bethe lattice are examined. Finally, in the conclusion we mention other areas where these ideas are useful.

Throughout the paper, the Bethe lattice with coordination 3 and the simple triangular cactus will be used as examples. The only exception is the discussion of the vibrational spectrum of
a Bethe lattice which is illustrated with a $z=4$ tetrahedral Bethe lattice.

2. HISTORICAL

Following the mean field theory of magnetism\(^1\) in which an effective field is placed on a single site, the Bethe-Peierls approximation\(^2,3\) was introduced in the mid 1930's to describe crystalline alloys (like $\beta$-brass) or Ising models. This work will be described in section 3. It was pointed out later that this approximation becomes exact on the Bethe lattice. While this observation is generally credited to Domb,\(^4\) it was in fact first used by Kurata, Kikuchi, and Watari\(^5\) as pointed out by Domb.\(^6\)

In their paper, Kurata, Kikuchi, and Watari\(^5\) describe a "dendritic structure" and give a sketch for a Bethe lattice with four neighbors ($z=4$) (see Fig. 7 of reference 5). To quote from this paper "The unique feature of the lattice is that it does not include any closed cells such as triangle, square, etc." also "Therefore, such a lattice is the one for which Bethe's method gives rigorous results, and might be called the 'Bethe lattice'." We show a sketch of a $z=3$ Bethe lattice in Fig. 1. We will refer to this as a pseudo-lattice because it cannot be embedded in any real lattice. It has a very large number of surface atoms. If we start with a single atom, connect up to $z$ neighbors and keep going for $n$ generations, the total number of atoms $N$ is given by

$$N = 1 + z + z(z-1) + \ldots + z(z-1)^{n-1}$$

$$(1)$$

$$= [z(z-1)^n - 2]/(z-2)$$

The number of atoms on the surface $N_s$ is

$$N_s = z(z-1)^{n-1}$$

(2)

Fig. 1. Showing a piece of a $z=3$ Bethe lattice.
Therefore for large $n$

$$\frac{N_s}{N} = \frac{(z-2)}{(z-1)}$$  \hspace{1cm} (3)

and a finite fraction of the total atoms are on the surface (for $z = 3$, one-half of the atoms are on the surface). In real lattices

$$\frac{N_s}{N} \sim N^{-1/d}$$  \hspace{1cm} (4)

for large $N$ which goes to zero in the thermodynamic limit for $d > 1$. Of course for $z = 2$ the Bethe lattice is a linear chain which is a real lattice.

It is important to distinguish between the Bethe lattice and the Cayley tree. The Bethe lattice is an infinite Cayley tree. Cayley trees are interesting mathematically\(^8\) but have no application to real lattices that I am aware of. For example there are surface states whose amplitude increases for a certain distance inside and is then zero. The density of states is pathological, consisting of delta functions which are nowhere continuous but have dense support.\(^9\) In the Bethe lattice one looks for properties that are unaffected by the surface although this cannot be avoided in all cases.\(^10\)

Another useful viewpoint that we will adopt is that the Bethe lattice represents the summation of the tree diagrams on a real lattice.

There are other pseudo lattices that are sometimes useful. The simplest is the Husimi cactus. This term was first introduced as "Husimi tree" by Riddell and Uhlenbeck,\(^12\) based on the work of Husimi\(^13\) although Husimi does not use either of the terms tree or cactus. From reference 12 we have the definition "A Husimi tree is a connected graph in which no line lies on more than one cycle." They were first used in connection with Mayer cluster expansions for gases.\(^12,13\) In Fig. 2 we show a simple triangular cactus. [This appears in Fisher and Essam\(^14\) (see Fig. 1) and may be the first use of the word "cactus" for such diagrams.] The Husimi cactus can be useful in investigating the effects of small rings. However it contains many fewer small rings than a real lattice and it must be used with caution. Some physical properties on the triangular cactus are similar to those on a very open lattice with triangles (e.g. the Kagomé lattice shown in Fig. 3). It must be emphasized that Husimi cacti are pseudo lattices and as such are in the same class as Bethe lattices with macroscopic surfaces etc.

3. THE BETHE–PEIERLS APPROXIMATION

The Bethe approximation or Bethe–Peierls approximation\(^2,3\) was originally applied to alloys where the variable $\sigma = \pm 1$ depending on whether a site was occupied by an A or B atom. Peierls\(^3\) pointed
Fig. 2. Showing a piece of a triangular cactus.

Fig. 3. Showing a piece of the Kagomé lattice.

out the equivalence to the problem of ferromagnetism and we shall use that language here. The treatment follows that of Domb. Note that this corresponds to Bethe's first approximation.

The system is described by an Ising model

\[ H = J \sum_{<ij>} \sigma_i \sigma_j - h \sum_i \sigma_i \] , where \( \sigma_i = \pm 1 \]

(5)
where the brackets $<>$ indicate a sum over nearest neighbors pairs only. The Bethe approximation considers a cluster of $(z+1)$ sites as shown in Fig. 4 described by a Hamiltonian $H_{c1}$ given by

$$H_{c1} = -J \sum_{i=1}^{z} \sigma_i \sigma_{i+1} - h \sigma_0 - h_1 \sum_{i=1}^{z} \sigma_i$$  \hspace{1cm} (6)$$

The yet to be determined field $h_1$ is there to simulate that part of the system outside the cluster. The partition function $z_{c1}$ is given by

$$z_{c1} = \sum_{r=0}^{\frac{z}{2}} z_{C_2} \left[ \mu^{-k}(\mu_1 \nu)^{-k} z^{-r} + \mu^{k}(\mu_1 / \nu)^{-k} z^{-r} \right]$$  \hspace{1cm} (7)$$

where

$$\mu = \exp(-2\beta h)$$

$$\mu_1 = \exp(-2\beta h_1)$$

$$\nu = \exp(-2\beta J)$$

Therefore

$$z_{c1} = \mu^{-\frac{k}{2}}(\mu_1 - 2 + \mu_1)^{-\frac{k}{2}} z + \mu^{\frac{k}{2}}(\mu_1 / \nu - 2 + \mu_1)^{-\frac{k}{2}} z$$  \hspace{1cm} (8)$$

Now the key step is to demand that the magnetization [i.e. $<\sigma>$] is the same on the central site and the neighbors. This leads to

$$\frac{\partial \ln z_{c1}}{\partial \mu} = \frac{1}{z} \frac{\partial \ln z_{c1}}{\partial \mu_1}$$  \hspace{1cm} (8)$$

Fig. 4. Showing a cluster with a central atom and $z = 3$ neighbors. The exchange couplings $J$ and the fields $h, h_1$ are indicated.
or
\[ \frac{\mu_1}{\mu} = \frac{\mu_1 + \nu}{\mu_1 + \nu} \] (8a)

In the absence of an external field \((h = 0, \mu = 1)\), this maybe rewritten
\[ \frac{\tanh[\beta h_1/(z-1)]}{\tanh[\beta h_1]} = \tanh[\beta J] \] (8b)

There is a solution with \(h_1 > 0\) for \(\beta > \beta_c\). As the critical point is approached, \(h_1 \to 0\) and
\[ \frac{\beta c h_1/(z-1)}{\beta c h_1} = \tanh[\beta_c J] \] (9)
or
\[ \tanh[\beta_c J] = 1/(z-1) \] (9a)

Thus the critical temperature \(T_c = \beta_c^{-1}\) is nonzero for all \(z > 2\). In one-dimension \(T_c = 0\). All thermodynamic quantities of interest can be derived from this approximation. We will concentrate our attention on \(\beta_c\).

In order to see why the Bethe approximation becomes exact on the Bethe lattice we consider a single bond in the Bethe lattice as shown in Fig. 5. If all effects to the right of site \(j\) are represented by a field \(h_j\), then we may write

\[ \text{Fig. 5. Showing a piece of a } z=3 \text{ Bethe lattice. The field } h_j \text{ represents the sum of all effects to the right of site } j \text{ as indicated by the dashed lines.} \]
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\[ \text{Tr} e^{\beta J i_j \sigma_j} e^{\beta h_i \sigma_i} e^{\beta h_j \sigma_j} \]
\[ = A e^{2 \beta h_j} \]
\[ j \]
from which we may show that
\[ 2 \beta h_j \]
\[ = \frac{\cosh(\beta J + \beta h_j)}{\cosh(\beta J - \beta h_j)} \]

Near \( \beta_c \), \( h_1 \) is small and therefore \( h_2 \) is also small and given by
\[ h_2' = (\tanh \beta J) \cdot h_1 \]

However there are a total of \((z-1)\) bonds reaching \(i\) from the right; all of which are identical to the one we have considered so the equivalent effective field to \( h_1 \) in the next generation is \( h_2 \) and is given by
\[ h_2 = [(z-1) \cdot \tanh \beta J] h_1 \]

Repeating this process we find that
\[ h_{n+1} = [(z-1) \cdot \tanh \beta J]^n h_1 \]

Clearly the critical temperature is given by (9a) when \( h_{n+1} = h_1 \), reproducing (9a).

We see that the tree like structure has the special property that any two points are connected by a single path and information can only flow from one point to another along this path. This is very different from real lattices.

Yet another way to solve the Hamiltonian (5) on a Bethe lattice is to define new variables \( \tau_{ij} \) associated with the bond connecting the spins \( i \) and \( j \)

\[ \tau_{ij} = \sigma_i \sigma_j \]

The \( \tau_{ij} \) are all independent because there are no rings of bonds. In the absence of an external field, the Hamiltonian is just a sum of non-interacting Hamiltonians.

\[ H = \sum_{\langle ij \rangle} H_{ij} \]

where
\[ H_{ij} = -J \tau_{ij} \]
Now for $\beta < \beta_c$, the susceptibility may be written

$$\beta \chi = \sum_{i,n} \langle \sigma_i \sigma_{i+n} \rangle$$

(18)

where the brackets $\langle \cdots \rangle$ denote thermal average. Now

$$\langle \sigma_i \sigma_{i+n} \rangle = \langle \sigma_{i+1} \sigma_{i+1} \sigma_{i+2} \cdots \sigma_{i+n} \rangle$$

$$= \langle \sigma_{i} \sigma_{i+1} \rangle \langle \sigma_{i} \sigma_{i+2} \rangle \cdots \langle \sigma_{i+n-1} \sigma_{i+n} \rangle$$

$$= [\tanh \beta J]^n$$

(19)

as $\langle \sigma_i \sigma_j \rangle = \tanh \beta J$.

$$\therefore \beta \chi = 1 + z \tanh \beta J + z(z-1)\tanh 2\beta J + \cdots$$

$$= [1 + \tanh \beta J]/[1 - (z-1)\tanh \beta J]$$

(20)

The susceptibility diverges at the critical temperature given by

$$(z-1)\tanh \beta_c J = 1$$

(9a)

In working out $\langle \sigma_i \sigma_{i+1} \rangle$, the side groups are irrelevant because each bond behaves independently. Note that the result (20) gives the solution to the linear chain with $z = 2$, that is usually calculated with a transfer matrix.\textsuperscript{15}

A simple generalization of the method just outlined also solves the case where there is a distribution of exchanges $J$ given by $P(J)$. The bonds still behave independently but the susceptibility is given by (20) with $\tanh \beta J$ replaced by

$$\tanh \beta J = \int P(J) \tanh \beta J dJ$$

(21)

and $\beta_c$ is given by

$$(z-1)\tanh \beta_c J = 1$$

(22)

For the special case of the dilute bond Ising model

$$P(J') = (1-p)\delta(J') + p\delta(J-J')$$

and (22) becomes

$$p(z-1)\tanh \beta_c J = 1$$

(23)
This phase boundary is shown in Fig. 6. It can be seen that the ferromagnetic phase does not exist for \( p < p_c = 1/(z-1) \). This is the percolation concentration.\(^{16}\) Phase transitions only occur in infinite systems. For \( p < p_c \), the systems breaks up into many finite regions of interacting spins.

In order to solve the \( S > \frac{1}{2} \) Ising model on the Bethe lattice, a more complex effective field would have to be used on the \( z \) outer atoms of the cluster shown in Fig. 4. For \( S = 1 \) this would have the form

\[
-h_1 \sum_i S_i^z - h_1' \sum_i (S_i^z)^2
\]

\[(24)\]

Finally in this section we examine the \( S = \frac{1}{2} \) Ising model on the triangular cactus\(^4,17\) of Fig. 2. We represent the sum of all effects to the right by fields \( h_1 \) as shown in Fig. 7. We write

\[
\frac{\beta J (\sigma_i^z \sigma_j^z \sigma_j \sigma_k^z \sigma_k^z)}{\prod_{j,k} e^{h_i} e^{h_j} e^{h_k}} = A e^{h_1} e^{h_2} \sigma_i^z \sigma_j^z \sigma_k^z
\]

\[(25)\]

which leads to

\[
2 \beta h_2 = \frac{e^{3 \beta J} e^{-2 \beta h_1} - e^{-2 \beta h_1}}{e^{3 \beta J} e^{-2 \beta h_1} - 2 e^{-2 \beta h_1} - e^{-2 \beta h_1}}
\]

\[(26)\]

Near the critical point \( \beta_c \) we expect \( h_1 \) and \( h_2 \) to be small, so that

\[
h_2 = 2 \left( \frac{e^{3 \beta J} - 1}{e^{6 \beta J} + 3} \right) h_1
\]

\[(27)\]

Fig. 6. Showing the phase boundary for the dilute bond Ising ferromagnet on the \( z = 3 \) Bethe lattice.
Fig. 7. Showing the cluster used to solve the Ising model on the triangular cactus.

and therefore

\[ h_{n+1} = 2 \left( \frac{e^{-4\beta J}}{e^{4\beta J} + 3} \right)^n h_1 \]  

(28)

Using the same logic as for the Bethe lattice, \( \beta_c \) is given by

\[ 2 \left( \frac{4\beta_c J}{e^{4\beta_c J}} \right) = 1 \]

or

\[ \exp(4\beta_c J) = 5 \]  

(29)

The idea of using an effective field should now be clear. It can be used on any pseudo-lattice if the unit is chosen appropriately. The effective field may be regarded as the sum of all diagrams in a high temperature expansion that start at a particular point. These diagrams are constructed by starting at a point, doing a random walk on the lattice and returning to that point. We will have more to say about them later.
4. PERCOLATION

Suppose the bonds in the lattice are regarded as pipes through which liquid can flow and each bond is either open or closed with probability $p$ or $q = 1-p$ respectively. This defines the bond percolation problem. There is a similar site percolation problem. Percolation is said to occur when a liquid can flow in the pipes and penetrate the whole system.

We define a generating function

$$P(p, q) = \sum_{r, s} C_{rs} p^r q^s$$

(30)

where $C_{rs}$ is the probability of a given bond belonging to a cluster of $r$ occupied bonds surrounded by $s$ empty bonds that isolate it from the rest of the medium. The mean cluster size $m$ is given by

$$m = \sum_{r, s} C_{rs} r p^r q^s = p \frac{\partial P(p, q)}{\partial p}$$

(31)

where $p$ and $q$ are treated as independent variables.

For the Bethe lattice we assign $\Sigma$ to the sum of all probabilities associated with bonds arriving at a vertex as shown in Fig. 8. This is analogous to the effective field $h_1$ used in the previous section (see Fig. 5). We see that

$$\Sigma = (q + p\Sigma)^2$$

(32)

and

$$P = q + p\Sigma^2$$

(33)

Fig. 8. Showing how $\Sigma$ represents the sum of all probabilities associated with the dashed lines.
Therefore from (31) and (33),
\[ m = p(\Sigma^2 + 2p \frac{\partial \Sigma}{\partial p}) \]
(34)
and from (32)
\[ \frac{\partial \Sigma}{\partial p} = 2(q+p\Sigma)(\Sigma + p \frac{\partial \Sigma}{\partial p}) \]
(35)
so that
\[ m = p \left[ \Sigma^2 + \frac{4p\Sigma^2(q+p\Sigma)}{1-2(q+p\Sigma)p} \right] \]
(36)
Finally we put \( q = 1-p \), so that \( \Sigma = 1 \) and find
\[ m = p[1+2p]/[1-2p] \]
(37)
This gives the mean cluster size for \( p > p_c \), where \( p_c \) is where \( m \) diverges i.e.
\[ p_c = \frac{1}{2} \]
(38)
as found previously in section 3 with \( z = 3 \). The mean cluster size is analogous to the susceptibility in magnetism.

The same technique can be used to find the mean cluster size and \( p_c \) for the triangular cactus of Fig. 2. We now define \( \Sigma \) as the sum of all probabilities associated with the dashed lines as shown in Fig. 9. Then
\[ \Sigma = q^2 + 2pq^2 \Sigma + (3p^2q + p^3) \Sigma^2 \]
(39)

Fig. 9. Showing how \( \Sigma \) represents the sum of all probabilities associated with the dashed lines.
and \( P = \mathbb{E}^2 \)

As before

\[
m = p \frac{\partial P}{\partial q} = 2p \mathbb{E} \frac{\partial \mathbb{E}}{\partial p}
\]

when \( q = 1-p \)

\[
m = 2p[2+2p-p^2]/[1-2(p+p^2-p^3)]
\]

(40)

The percolation concentration is given by

\[
l = 2(p_c + p_c^2 + p_c^3)
\]

which leads to a single root for \( 0 < p_c < 1 \) given by

\[
p_c = 0.4030
\]

5. TIGHT BINDING HAMILTONIANS

Tight binding Hamiltonians can also be solved on the Bethe lattice. We illustrate this by using the simplest hopping Hamiltonian:

\[
H = V \sum_{\langle i,j \rangle} |i><j|
\]

(41)

where \( |i> \) is an s-like state associated with site \( i \). It is convenient to define a Green function \( G_{ij} \) by

\[
G_{ij} = \sum_k \frac{|i><k|<k|j>}{E - E_k}
\]

(42)

where \( E_k \) are the eigenvalues and \( |k> \) the eigenstates. The label \( k \) merely labels the eigenstates and has nothing to do with momentum as there is no translational invariance in the usual sense and hence no Bloch theorem. The density of states is given by

\[
\rho(E) = -\frac{1}{\pi N} \text{Im} \sum_i G_{ii}
\]

(43)

where \( N \) is the total number of atoms. The Green function obeys the equation of motion

\[
E G_{ij} = \delta_{ij} + V \sum_{i'} G_{i'j}
\]

(44)

where \( i' \) is a near neighbor of \( i \). At large \( E \) we may write
\[ G_{i1} = \sum_{n=0}^{\infty} \frac{\langle i| H^n |1 \rangle}{E^{n+1}} \]  

(45)

and the \( n \)th term in the summation may be regarded as a diagram that starts at site \( i \) and returns to site \( i \) after an \( n \) step walk. It is the sum of all such diagrams that will represent by an effective field.

We first solve the problem in the spirit of the Bethe approximation. Using a cluster of \( (z+1) \) atoms as shown in Fig. 10.

\[ H_{c1} = V \sum_{i=1}^{z} [\langle 0<i \rangle + \langle i<0 \rangle] + h \sum_{i=1}^{z} \langle i<i \rangle \]  

(46)

where 0 is the central site and there are \( z \) neighbors. We find that

\[ \begin{align*}
E \, G_{00} &= 1 + z \, V \, G_{10} \\
(E-h)G_{10} &= V \, G_{00} \\
(E-h)G_{11} &= 1 + V \, G_{10}
\end{align*} \]  

(47)

Solving these equations

\[ \begin{align*}
G_{00} &= \frac{1}{E - \frac{zV^2}{E-h}} \\
G_{11} &= \frac{\left( E - (z-1)V^2 \right) \left( \frac{1}{E-h} \right)}{E - \frac{zV^2}{E-h}} \end{align*} \]  

(49)

Fig. 10. Showing how an effective field \( h \) can be used to solve the tight binding Hamiltonian on the Bethe lattice. The hopping integrals are \( V \) and the effective field is \( h \).
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In the spirit of the Bethe approximation we demand self-consistency

\[ G_{00} = G_{11} \] (50)

which leads to

\[ h(E-h) = (z-1)V^2 \] (51)

or

\[ 2h = E \pm \sqrt{E^2 - 4(z-1)V^2} \] (52)

and hence

\[ G_{00} = \frac{2(z-1)}{(z-2)E \pm z \sqrt{E^2 - 4(z-1)V^2}} \] (53)

and \( \rho = -1/\pi \, \Im \, G_{00} \)

\[ \rho = \frac{z}{2\pi} \, \frac{\sqrt{4(z-1)V^2 - E^2}}{(zV)^2 - E^2}, \] for \( |E| \leq 2\sqrt{z-1} \, V \). (54)

where we choose the + sign as \( \rho(\omega) \geq 0 \). The density of states integrates to one as we would expect and is shown in Fig. 11 for \( z = 3 \). The band is defined by

\[ |E| \leq 2\sqrt{z-1} \, V \] (55)

![Graph of density of states](image)

Fig. 11. Showing the density of states for the simple tight binding Hamiltonian on the Bethe lattice with \( z = 3 \).
which is narrower than the expected band from the theorem of Perron.18,19

\[ |E| \leq zV. \]  

(56)

This is because "long wavelength" excitations cannot exist due to the pathological surface. The Green function has two sheets in the complex E plane; one physical and one unphysical. They touch at the two points that define the band edge (55). It might be suspected from (54) that there are also simple poles at \( E = \pm zV \). In fact, the residue at these poles is zero on the physical sheet. [Note however that for a finite Cayley tree there would be single states at \( E = \pm zV \), and \( N-2 \) states in the band]. The physical sheet can be selected pragmatically by noting that

1) \( \text{Im } G_{00} \leq 0 \) for \( |E| < zV \)

2) \( G_{00} = 1/E \) for large \( |E| \).

This derivation follows the spirit of the Bethe approximation. An easier approach is the following. Let h be defined as in Fig. 12 and \( a_i \) and \( a_j \) be the amplitudes of the wavefunctions associated with sites i and j. Then

\[ E a_i = V a_j + \text{etc.} \]

\[ (E-h) a_j = V a_i \]  

(57)

where "etc." is everything to the left of site i. Thus

\[ \left( E - \frac{V^2}{E-h} \right) a_i = \text{etc.} \]  

(58)

Fig. 12. Showing how h represents the sum of all effects to the right of site j.
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We may use this equation to define another effective field \( h' \)

\[
h' = \frac{V^2}{E-h} \tag{59}
\]

and \((z-1)\) contributions \( h' \) lead to \( h \) if site \( i \) and \( j \) are equivalent (i.e. well away from the surface)

\[
\therefore h = (z-1)h' = \frac{(z-1)V^2}{E-h} \tag{51}
\]

which is the result obtained previously. A little thought will convince the reader that

\[
G_{00} = \frac{1}{E-zh'} = \frac{1}{E - \left(\frac{z}{z-1}\right)h} \tag{60}
\]

from which the results (53) and (54) are again obtained.

This latter method can be used to solve the tight binding Hamiltonian (41) on the triangular cactus (see Fig. 2). Using notation that should be clear from the previous discussion, and is indicated in Fig. 13.

\[
\begin{align*}
E a_i &= V(a_j + a_k) + \text{etc.} \\
(E-h)a_j &= V(a_k + a_i) \\
(E-h)a_k &= V(a_i + a_j)
\end{align*} \tag{61}
\]

Fig. 13. Showing the cluster used in the solution of the tight binding Hamiltonian on the triangular cactus.
From which we find by eliminating $a_j$ and $a_k$,

$$\left( E - \frac{2V^2}{E-h-V} \right) a_i = \text{etc.} \quad (62)$$

so that the equation that determines $h$ is

$$h = \frac{2V^2}{E-h-V} \quad (63)$$

and $G_{00}$ is given by

$$G_{00} = \frac{1}{E-2h} \quad (64)$$

From (63) and (64) we find

$$G_{00} = \frac{\sqrt{\frac{(E-V)^2 - 8V^2}{9V^2 - (E-V)^2}}}{\pi} \quad (65)$$

and

$$\rho = -\frac{1}{\pi} \text{Im} G_{00}$$

$$= \pm \frac{1}{\pi} \sqrt{\frac{8V^2 - (E-V)^2}{9V^2 - (E-V)^2}} \quad (66)$$

Choosing the $+$ sign, this defines a band between

$$1 - \sqrt{8} \leq E/V \leq 1 + \sqrt{8}$$

as shown in Fig. 14.

![Graph](image)

Fig. 14. Showing the density of states for the triangular cactus. The band has weight $2/3$ and the delta function has weight $1/3$. 
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The integrated weight in (66) is only 2/3. Careful examination shows that there is a pole on the physical sheet at \( E = -2V \) with residue 1/3. This is also shown in Fig. 14.

This pole is a little unusual and can be attributed to the triangles. It would not occur in a simple cactus made up of polygons with more than three sides. We note that as every site in the triangular cactus has 4 neighbors, the theorem of Peron (equation 56) is obeyed.

The delta function is formed from states where the sum of the amplitudes associated with the three states on each triangle is zero. This gives \( E = -2V \) and 1 constraint per triangle. If there are \( N \) atoms, this means there are \( N - 2N/3 = N/3 \) independent ways to form this state. Hence the delta function contains \( 1/3 \) of all the states.

5. PHONONS

Up until now all the problems that we have considered have involved only the topology or connectivity of the pseudo-lattice. It has only been important what is connected to what, and not what the angles were. Also we have not attempted to define distance in the Bethe lattice. In fact it is impossible to define distance in any meaningful way so that although we have been able to calculate densities of states, it is not possible to calculate the neutron scattering as this involves \( k \) that is the conjugate variable to a distance.

However it is possible to introduce geometrical aspects into the Bethe lattice in some cases and we will illustrate this.

We consider the Born model\(^{20}\) in which the potential energy is

\[
V = \frac{(\alpha - \beta)}{2} \left( \sum_{i<j} \left( \hat{\mathbf{r}}_{ij} \right) \cdot \hat{\mathbf{r}}_{ij} \right)^2 + \frac{\beta}{2} \sum_{i<j} \left( \hat{\mathbf{u}}_i \cdot \hat{\mathbf{u}}_j \right)^2
\]

(67)

and where \( \hat{\mathbf{u}}_i \) is the (vector) displacement of atom \( i \), \( \hat{\mathbf{r}}_{ij} \) is a unit vector between nearest neighbor atoms and \( \alpha, \beta \) are the central, non-central force constants respectively. If the mass of all the atoms is \( M \), then the equation of motion is

\[
M \frac{\partial^2 \hat{\mathbf{u}}_i}{\partial t^2} = -\frac{\partial V}{\partial \hat{\mathbf{u}}_i}
\]

(68)

We consider a tetrahedral Bethe lattice with \( z = 4 \) and the tetrahedral angle \( \cos^{-1}(-1/3) = 108^\circ \) between all adjacent bonds. It is necessary to introduce two fields \( h^n \) and \( h^t \) to represent
the perpendicular and transverse motion of all the dashed bonds
to the right of \( j \) as shown in Fig. 15. For motion parallel to \( \vec{r}_{ij} \)
we have

\[
(M_0^2 - \alpha) u_j'' = h'' u_j'' - \alpha u_i''
\]
\[
(M_0^2 - \alpha) u_i'' = -\alpha u_j'' + \text{etc.}
\]

where as before the "etc." represents everything to the left of
atom \( i \). Equations (69) can be rewritten

\[
\left[ \frac{M_0^2 - \alpha}{M_0^2 - \alpha - h''} \right] u_i'' = \text{etc.}
\]

so that we can define a new effective field

\[
h_1'' = \alpha + \frac{\alpha}{M_0^2 - \alpha - h''}
\]

Similarly for motion perpendicular to \( \vec{r}_{ij} \), we have

\[
h_1^i = \beta + \frac{\beta}{M_0^2 - \beta - h^i}
\]

At site \( i \), the contributions from 3 bonds like \( ij \) must be added
together in order to get a closed set of equations for the
effective fields. A little geometry shows that

![Diagram illustrating effective fields](image)

**Fig. 15.** Showing the effective fields \( h'' \), \( h^i \) used to solve for
the vibrations of a Bethe lattice.
BETHE LATTICES

$$3h^{''} = h^{''}_1 + 8h^+_1$$

$$3h^4 = 4h^{''}_1 + 5h^+_1$$

and the diagonal displacement-displacement Green function $G$ is given by

$$G = \frac{1}{M^2 - \frac{4}{3}(h^{''}_1 + 2h^+_1)}$$

(74)

By eliminating all the effective fields from these equations we are left with a single equation for $G$,

$$\pm \frac{2}{3} \sqrt{1 + (2\alpha G)^2} \pm \frac{4}{3} \sqrt{1 + (2\beta G)^2} = 1 + G[M\omega^2 - \frac{4}{3} (\alpha + 2\beta)]$$

(75)

This is more convenient than solving for the fields and then finding $G$. We could have proceeded in this way for the tight binding Hamiltonian considered previously.

There are 4 solutions to equation 75, only one of which has the property that $\text{Im } G \leq 0$ for all frequencies. This is the physical solution and is shown in Fig. 16 with $\alpha/\beta = 2/11$. It is

![Graph showing density of states](image)

**Fig. 16.** Showing the density of states for the Born model for the Bethe lattice (solid line) and the diamond lattice (dashed line). The maximum frequency $\omega_{\text{max}}$ is given by $M\omega_{\text{max}}^2 = 8/3 (\alpha + 2\beta)$. 
compared with a density of states for the diamond cubic structure calculated in the usual way for a real lattice by obtaining the dispersion relations $\omega(k)$ and doing a Brillouin zone integration. It can be seen that many of the features in the spectra are due to the tetrahedral coordination and not to the long range structural order. It should not surprise us that the sound waves are pathological—indeed they do not exist as there are no modes at the lowest frequencies.

7. CONCLUSION

We have tried to give a feeling for what can be done with Bethe lattices and the rather special techniques used. In all cases the results for general $z$ reduce to those of the linear chain for $z = 2$. The attraction of the Bethe lattice is that no approximations are made so that thermodynamic equalities are maintained, Green functions are Herglotz etc. The disadvantage is the unphysical nature of the lattice. Often in diagrammatic expansions on real lattices, the first (and often only) set of diagrams that can be summed are the tree diagrams that correspond to a Bethe lattice. The Bethe lattice forms a poor approximation to a lattice like fcc with many small rings but is much better for an open lattice like diamond.

The Bethe lattice has been used to study many other phenomena including localization,$^{22}$ random resistor networks,$^{23}$ and even trees.
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