A bond-diluted Ising ferromagnet with first- and second-neighbour interactions†

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Abstract. We study the effects of second-neighbour interactions on the critical temperature of the bond-diluted Ising model within the framework of the decorated Ising model. A suitable second-neighbour bond problem can be formulated and solved in the annealed limit. This limit is known to be quite reliable for Ising models without competing interactions. Plots of the critical-temperature and zero-temperature magnetisation versus concentration are given for various ratios of the second to first neighbour exchange coupling. When this ratio is very small, but finite, there is a tail in the phase boundary.

1. Introduction

Recent efforts to understand the behaviour of dilute magnetic systems in greater detail has resulted in considerable progress. The low-temperature excitations have been studied using neutron scattering and the effects of clusters clearly seen in Rb₂Mn₉₉Mg₁₋₉₉F₄ with $p \approx 0.54$ (Cowley et al 1977). The highly structured spectral response functions can be obtained very accurately by solving the linearised equations of motion for the disordered system using numerical techniques (Kirkpatrick and Harris 1975, Thorpe and Alben 1976). The dependence of the transition temperature upon concentration $p$ has also been studied for a number of systems, e.g. Mn₉₉Zn₁₋₉₉F₂ (Baker et al 1961). This system is particularly interesting because the second largest exchange cannot be completely neglected as it produces a 'tail' in the phase diagram of the kind sketched in figure 1. This has recently been discussed for Mn₉₉Zn₁₋₉₉F₂ using a Heisenberg Hamiltonian and employing CPA and CPA approximation schemes (McGurn and Tahir-Kheli 1978). Good agreement was obtained between the theory and experiment.

In order to obtain a better understanding of the general phenomena of tails in the phase diagrams we have set up a simple model that can be solved and shows the effect. The model is an extension of the random-bond annealed Ising model (Rapaport 1972, Thorpe and Beeman 1976). The solution of this model leads to a phase boundary that, in the case of the dilute ferromagnet, is very close to that in the quenched limit (i.e. within a per cent or so at all concentrations studied). Indeed we expect the quenched and annealed bond problems to lead to very similar thermodynamic behaviour as long as there are no exchange interactions of opposite sign. Of course this is not true in site problems where the quenched and annealed versions lead to very different thermodynamic behaviour.

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Figure 1. A sketch of the phase diagram expected for a site-diluted ferromagnet with first and second-neighbour exchanges on a square lattice. The percolation concentrations for the first and second-neighbours are taken from Essam (1972).

In order to have a model with second-neighbour interactions that is tractable in the annealed limit, we have set up a model on a decorated lattice. A skeleton lattice with equivalent sites (e.g. square net, honeycomb, simple cubic etc.) is decorated; that is an extra site is placed at the centre of each bond as shown in figure 2. The nearest-neighbour exchange is $J$ and the next-nearest exchange is $I$. We show that this model can be solved for general probability distributions of $J$ and $I$ and then concentrate our attention on the dilute ferromagnet. The method of solution involves the introduction of chemical potentials and is similar to Thorpe and Beeman (1976). The model does indeed display the qualitative features that we expect (e.g. a phase diagram similar to the sketch in figure 1).

Figure 2. (a) A sketch of a single decorated bond showing the notation used; $J_1^{(1)}$ and $J_2^{(1)}$ are the nearest-neighbour exchanges and $I_r$ is the second-neighbour exchange. Sites on the skeleton lattice are represented by closed circles. The open circles are the decoration sites. (b) A piece of a decorated square lattice.
The dependence of the critical-temperature and zero-temperature magnetisation on bond concentration is obtained for various ratios of the first to second neighbour exchange couplings. As is to be expected, the second-neighbour exchanges are very important in determining the critical concentration, \( p_c \), and the properties of the system in the neighbourhood of \( p_c \). Second-neighbour interactions lower the critical concentration to a point which depends on the percolation properties of the lattice with second-neighbour interactions (Essam 1972).

The layout of this paper is as follows. In the next section we set up the Hamiltonian for the model and solve it for a general probability distribution. In \( \S 3 \), we study the special case of the dilute ferromagnet with second-neighbour interactions in some detail, paying particular attention to the phase diagram and the magnetisation at zero temperature. Two models are set up; in the first, the second-neighbour exchange is highly correlated with the first-neighbour exchange, while in the second it is independent. Qualitatively different kinds of phase diagrams are obtained for the two models. In the conclusion we make some general comments about this kind of model.

2. General solution

We study the random-bond problem for decorated lattices. A decorated square lattice is shown in figure 2. The nearest-neighbour exchange can take on a series of values \( J_p \). The indicator function \( f_p \) is 1 if the exchange is \( J_p \) and zero if the exchange is \( J_l \) where \( l \neq p \). The second-nearest-neighbour exchange can take on a series of values \( I_r \) and has an associated indicator function \( g_r \). The Hamiltonian \( H \) is a sum of bond Hamiltonians \( H_{ij} \) where

\[
H = \sum_{(ij)} H_{ij}
\]

and

\[
H_{12} = -\sum_p J_p^{(1)} f_p^{(1)} \sigma_1 \sigma_0 - \sum q J_q^{(2)} f_q^{(2)} \sigma_2 \sigma_0 - \sum r I_r g_r \sigma_1 \sigma_2 - \sum_{pq} C_{pq} f_p^{(1)} f_q^{(2)} g_r
\]

The superscripts 1 and 2 refer to the left and right hand sides of the decorated bond (see figure 2a) and \( \sigma_k = \pm 1 \). The chemical potentials \( C_{pq} \) couple to the product of the three indicator functions associated with each bond. The indicator functions obey the operator identities

\[
\sum_p f_p^{(i)} = 1, \quad i = 1, 2
\]

\[
\sum_r g_r = 1
\]

which say that one and only one of the possible exchanges is realised. We will do the algebra for a discrete distribution of exchanges and go to the continuum limit at the end of the calculation in the fashion of Thorpe and Beeman (1976). The chemical potentials are chosen to fix

\[
\langle f_p^{(1)} f_q^{(2)} g_r \rangle
\]

for each bond. In the continuum limit, the expression (5) becomes the joint probability function for the three exchanges on the decorated bond

\[
P(f_p^{(1)}, f_q^{(2)}, I_r).
\]

(5a)
The grand partition function $\Xi$ involves a trace over both the spin variables and the indicator functions:

$$\Xi = \sum_{\{f^{(1)}_p f^{(2)}_q\}} \sum_{\{\sigma_1 \sigma_2\}} \text{Tr} \exp(-\beta H).$$  \hspace{1cm} (6)

Doing the partial trace over the $f^{(0)}_p$, $g$, and the decoration spin for the single decorated bond shown in figure 2a, we obtain

$$\sum_{pq} 2 \cosh[\beta(J^{(1)}_p \sigma_1 + J^{(2)}_q \sigma_2)] \exp[\beta I_\sigma \sigma_3] \exp[\beta C_{pqr}] = A \exp(K \sigma_1 \sigma_2)$$  \hspace{1cm} (7)

where

$$\exp 2K = \sum_{pq} \cosh[\beta(J^{(1)}_p + J^{(2)}_q)] \exp[\beta I_\sigma] \exp[\beta C_{pqr}]$$  \hspace{1cm} \sum_{pq} \cosh[\beta(J^{(1)}_p - J^{(2)}_q)] \exp[-\beta I_\sigma] \exp[\beta C_{pqr}],$$  \hspace{1cm} (8)

and

$$A^2 = (\sum_{pq} 2 \cosh[\beta(J^{(1)}_p + J^{(2)}_q)] \exp[\beta I_\sigma] \exp[\beta C_{pqr}])$$

$$\times (\sum_{pq} 2 \cosh[\beta(J^{(1)}_p - J^{(2)}_q)] \exp[-\beta I_\sigma] \exp[\beta C_{pqr}]).$$  \hspace{1cm} (9)

The right hand side of equation (7) describes a single bond in a nearest-neighbour regular Ising model isomorphic to the skeleton lattice. Doing the partial traces as above for all the decorated bonds, we can write

$$\Xi = A^{Nz/2} z(K)$$  \hspace{1cm} (10)

where $z(K)$ is the partition function for a regular Ising model on the skeleton lattice (Onsager 1944) with interaction parameter $K$ and $N$ sites each with 2 neighbours. Notice that our result in equation (10) is useful for any lattice when the regular Ising model can be solved. This allows us to apply it directly to two-dimensional lattices. Good numerical results for the critical parameters exist in three dimensions.

The chemical potentials $C_{pqr}$ can be obtained from

$$(Nz/2) \langle f^{(1)}_p f^{(2)}_q g \rangle = \frac{\partial \ln \Xi}{\partial C_{pqr}}$$

$$= Nz/2 \left[ \frac{\partial \ln A}{\partial C_{pqr}} + \epsilon(K) \frac{\partial K}{\partial C_{pqr}} \right].$$  \hspace{1cm} (11)

Here $\epsilon(K) = \langle \sigma_1 \sigma_2 \rangle$ is the nearest-neighbour two-spin correlation function for the regular Ising model of equation (10). Using equations (8) and (9) we obtain from equation (11)

$$\langle f^{(1)}_p f^{(2)}_q g \rangle \sum_{pq} \cosh[\beta(J^{(1)}_p + J^{(2)}_q)] \exp[\beta I_\sigma] \exp[\beta C_{pqr}]$$

$$= \frac{1}{2}[(1 + \epsilon(K)) + (1 - \epsilon(K)) Q_{pqr} \exp(2K)] \cosh[\beta(J^{(1)}_p + J^{(2)}_q)]$$

$$\times \exp[\beta I_\sigma] \exp[\beta C_{pqr}],$$  \hspace{1cm} (12)

where

$$Q_{pqr} = \frac{\cosh[\beta(J^{(1)}_p - J^{(2)}_q)]}{\cosh[\beta(J^{(1)}_p + J^{(2)}_q)]} \exp[-2\beta I_\sigma].$$  \hspace{1cm} (13)
From equation (12) we obtain an implicit equation for the temperature mapping

$$\sum_{\rho \rho'} \frac{\langle f^{(11)}_{\rho} f^{(2)}_{\rho'} \rangle}{[1 + \epsilon(K)] \left[ 1 - \epsilon(K) \right] Q_{\rho \rho'} \exp(2K)} = \frac{1}{2}. \quad (14)$$

Going from a discrete distribution of exchange interactions to a continuous one; we go from a summation involving expression (5) to an integral involving (5a), so that our result (14) becomes

$$\int \int \int \frac{P(J_{p}, J_{q}, I_{r}) dJ_{p} dJ_{q} dI_{r}}{[1 + \epsilon(K)] \left[ 1 - \epsilon(K) \right] Q_{\rho \rho'} \exp(2K)} = \frac{1}{2}. \quad (14a)$$

We have suppressed the superscripts (1) and (2) on $J_{p}$ and $J_{q}$. It is clear that $J_{p}, J_{q}$ and $I_{r}$ are the three exchanges that are associated with the single decorated bond shown in figure 2a. The result (14a) maps the temperature $(k_{B}T)^{-1}$ of the random Ising model characterised by the joint probability function for each bond $P(J_{p}, J_{q}, I_{r})$ onto a regular Ising model with exchange parameter $J$ on the skeleton lattice at a temperature $(J/Kk_{B})$. The thermodynamic quantities of interest can be obtained after the manner of Thorpe and Beeman (1976).

3. Dilute ferromagnet

In order to obtain the ‘tailing effect’ in the phase diagram, it is necessary to choose a joint probability such that a weak second-neighbour exchange persists after the nearest-neighbour exchange has been severed. There are many ways of doing this—all of them somewhat arbitrary. We look at two models in this section for the dilute ferromagnet with second-neighbour exchange. The nearest-neighbour exchange is $J$ and the second-neighbour exchange $I$.

3.1. Model I

We assume that the second-neighbour bond in figure 2a is not severed until both the nearest-neighbour bonds have been cut. Thus the second-neighbour exchange is highly correlated with the first-neighbour exchange and this model is described by

$$P(J_{p}, J_{q}, I_{r}) = \left[ p^{2} \delta(J_{p} - J) \delta(J_{q} - J) + p(1 - p) \delta(J_{p} - J) \delta(J_{q}) + (1 - p)^{2} \delta(J_{p}) \delta(J_{q} - I) + (1 - p)^{2} \delta(J_{p}) \delta(J_{q}) \delta(I_{r}) \right]. \quad (15)$$

Equation (14a) then becomes

$$2p^{2} \left[ \frac{1}{1 + \epsilon(K)} + \frac{1}{1 - \epsilon(K)} \right] \exp(2K) \left[ \cosh 2BJ \right]^{-1} \exp \left[ -2\beta I \right]$$

$$+ \frac{4p(1 - p)}{\left[ 1 + \epsilon(K) \right] \left[ 1 - \epsilon(K) \right] \exp(2K) \exp \left[ -2\beta I \right]}$$

$$+ \frac{2(1 - p)^{2}}{\left[ 1 + \epsilon(K) \right] \left[ 1 - \epsilon(K) \right] \exp(2K)} = 1. \quad (16)$$

The critical temperature as a function of the nearest-neighbour bond concentration can be obtained from equation (16) by choosing $K$ to be the value $K_{s}$ of the skeleton lattice at its critical point. The values of $K_{s}$ and $\epsilon(K_{s})$ are given in Syosi and Miyazima (1966).
The critical concentration of nearest-neighbour bonds below which no long-range order exists, can then be obtained from equation (16). We find two critical concentrations:

1. If \( J \neq 0, I = 0 \)

\[
p_c^{(1)} = \left\{ \frac{1}{2} [1 + \epsilon(K_c)][1 - \exp(-2K_c)] \right\}^{1/2}
\]  

(17)

2. If \( I \neq 0 \)

\[
p_c^{(2)} = 1 - \left\{ 1 - \frac{1}{2} [1 + \epsilon(K_c)][1 - \exp(-2K_c)] \right\}^{1/2}.
\]  

(18)

Notice that \( p_c^{(1)} \) is the square root of the critical concentration of the single-bond annealed problem of Thorpe and Beeman (1976). Also we have the relationship

\[
p_c^{(2)} = 1 - \left[ 1 - (p_c^{(1)})^2 \right]^{1/2}.
\]  

(19)

![Figure 3](image)

**Figure 3.** The phase diagram for model I for various values of the ratio of the second to first neighbour exchange \( I/J \).

In figure 3 we show results for \( T_c(p)/T_c(1) \) versus \( p \), where \( p \) is the nearest-neighbour bond concentration, and \( T_c(p) = (k_B\beta(p))^{-1} \), where \( T_c(1) \) is given by

\[
cosh\left(\frac{2J}{k_B T_c(1)}\right) = \exp(-2K_c) \exp(-2I/k_B T_c(1)).
\]

From these results we can calculate the magnetisation in this model. The magnetisation of the skeleton lattice \( \langle \sigma_1 \rangle \) is obtained from Yang (1952)

\[
\langle \sigma_1 \rangle = \left[ 1 - \sinh^{-4}(2K) \right]^{1/8}
\]

where \( K \) is found from equation (16). The magnetisation at the decoration site can be calculated by adding a small external field \( h \) at each decoration site and proceeding as in
§ 2. Now the partial trace over the $f_{p}^{(0)} \sigma_{i}$ and decoration spin for a single bond yield an expression of the form

$$A \exp [K \sigma_{1} \sigma_{2}] \exp [H(\sigma_{1} + \sigma_{2})]$$

from which we find

$$\langle \sigma_{0} \rangle = \frac{\partial}{\partial h} \ln A + \langle \sigma_{1} \sigma_{2} \rangle \frac{\partial K}{\partial h} + 2\langle \sigma_{1} \rangle \frac{\partial H}{\partial h}. \tag{20}$$

In the limit $h \to 0$ we obtain

$$\langle \sigma_{0} \rangle = 2\langle \sigma_{1} \rangle \left\{ \frac{p^{2} \sinh [2\beta J]}{[1 + \epsilon(K)] \cosh [2\beta J] + [1 - \epsilon(K)] \exp [-2\beta I] \exp (2K)} \right.$$\n
$$\left. + \frac{2p(1 - p) \tanh [\beta J]}{[1 + \epsilon(K)] + [1 - \epsilon(K)] \exp [-2\beta I] \exp (2K)} \right\} \tag{21}$$

where $K$ and $\epsilon(K)$ are determined by equation (16). The magnetisation per site of the lattice is then

$$M = \frac{1}{3}(\langle \sigma_{1} \rangle + 2\langle \sigma_{0} \rangle). \tag{22}$$

In figure 4 we have plotted the zero-temperature magnetisations as a function of concentration. There are two sets of curves, one for $J \neq 0$, $I = 0$ and the other for $I \neq 0$, $J \neq 0$. The magnetisation vanishes at the critical concentration for each of the respective cases. For $I \neq 0$, $J \neq 0$ both $\langle \sigma_{0} \rangle$ and $\langle \sigma_{1} \rangle$ are independent of the ratio $I/J$ and are just a property of the lattice. It is similar to the probability of being in an infinite cluster $P(p)$ (Essam 1972). However it is a more complicated quantity because we are looking at a bond percolation problem, but the magnetisation measures the number of sites in the infinite cluster. In addition to this, we are dealing with an annealed rather than a quenched situation.

The quenched version of this problem would lead to very similar results as can be

![Figure 4](image-url)

**Figure 4.** The zero-temperature magnetisation of model 1 with $J \neq 0$, $I = 0$; and for $I \neq 0$, $J \neq 0$, for both the decoration sites, $\langle \sigma_{0} \rangle$, and the sites in the skeleton lattice, $\langle \sigma_{1} \rangle$, as indicated on the figure.
seen by considering \( T_c(1), p_c^{(1)} \) and \( p_c^{(2)} \). The transition temperature \( T_c(1) \) is the same in the annealed and quenched versions of the problems as there is no disorder and we have the decoration transformation (Syozi 1972). If \( p_c \) is the percolation concentration for the bond problem on the quenched skeleton lattice, then it is rather easy to show that
\[
p_c^{(1)} = \sqrt{p_c}
\]
and
\[
p_c^{(2)} = 1 - \sqrt{(1 - p_c)}.
\]

We see that equation (19) also holds in the quenched case and from table 1, we see that the annealed and quenched models give very similar results for all lattices. (For the square lattice they are actually identical—this can be traced back to the fact that the square net is self-dual, (Syozi 1972)).

<table>
<thead>
<tr>
<th>Lattice</th>
<th>( p_c^{(1)} ) annealed</th>
<th>( p_c^{(1)} ) quenched</th>
<th>( p_c^{(2)} ) annealed</th>
<th>( p_c^{(2)} ) quenched</th>
</tr>
</thead>
<tbody>
<tr>
<td>Honeycomb</td>
<td>0.8049</td>
<td>0.8079</td>
<td>0.4065</td>
<td>0.4107</td>
</tr>
<tr>
<td>Square net</td>
<td>0.7071</td>
<td>0.7071</td>
<td>0.2929</td>
<td>0.2929</td>
</tr>
<tr>
<td>Triangular net</td>
<td>0.5935</td>
<td>0.5893</td>
<td>0.1951</td>
<td>0.1921</td>
</tr>
<tr>
<td>Simple cubic</td>
<td>0.494</td>
<td>0.497</td>
<td>0.131</td>
<td>0.132</td>
</tr>
</tbody>
</table>

The most unsatisfactory feature of model I is that the first- and second-neighbour bonds are so highly correlated. To try and understand the effect of removing this correlation we look at another model.

3.2. Model II

In this model we allow the joint probability function \( P(J_p, J_q, I_r) \) for the bond in figure 2a to factorise so that each of the three bonds occurs independently with probability \( p \). Hence,
\[
P(J_p, J_q, I_r) = [p \delta(J_p - J) + (1 - p) \delta(J_p)] [p \delta(J_q - J) + (1 - p) \delta(J_q)]
\]
\[
\times [p \delta(I_r - I) + (1 - p) \delta(I_r)].
\]

Putting this into (13) and (14a), we get the temperature mapping
\[
p^3 = \frac{[1 + \epsilon(K)] + [1 - \epsilon(K)] \cosh 2BJ^{-1} \exp [-2\beta I] \exp (2K)}{p(1 - p^2)}
\]
\[
+ \frac{[1 + \epsilon(K)] + [1 - \epsilon(K)] \exp [-2\beta I] \exp (2K)}{p^2(1 - p)}
\]
\[
+ \frac{(1 - p^2)(1 - p)}{[1 + \epsilon(K)] + [1 - \epsilon(K)] \exp (2K)} = \frac{1}{2}.
\]
Figure 5. The phase diagram of model II for various values of the ratio of second to first neighbour interaction energies \( I/J \).

Substituting the critical parameters for the skeleton lattice, we obtain the phase diagram shown in Figure 5 for the square lattice for various values of \( I/J \). The most striking difference from model I is that there are three not two critical concentrations

1. If \( J \neq 0, I = 0 \)

\[
p_c^{(1)} = \left\{ \frac{1}{2} \left[ 1 + \epsilon(K_c) \right] \left[ 1 - \exp(-2K_c) \right] \right\}^{1/2}
\]

(27)

2. If \( J \neq 0, I \neq 0 \), then \( p_c^{(2)} \) is a solution of

\[
(p_c^{(2)})^3 - (p_c^{(2)})^2 - p_c^{(2)} = -\frac{1}{3} \left[ 1 + \epsilon(K_c) \right] \left[ 1 - \exp(-2K_c) \right]
\]

(28)

3. If \( J = 0, I \neq 0 \)

\[
p_c^{(3)} = \frac{1}{3} \left[ 1 + \epsilon(K_c) \right] \left[ (1 - \exp(-2K_c)) \right].
\]

(29)

The reason for this is as follows. For case 3 when \( J = 0 \) and \( I \neq 0 \), the system splits up into non-communicating clusters for concentrations below \( p_c^{(3)} \). The inclusion of a \( J \neq 0 \) allows these clusters to link up and percolate earlier at \( p_c^{(2)} \). Finally, when \( I = 0 \) we get \( p_c^{(1)} \) which is the same as in model I. For the quenched version of the problem \( p_c^{(1)}, p_c^{(2)} \) and \( p_c^{(3)} \) are given by:

\[
(p_c^{(1)})^3 - (p_c^{(1)})^2 - p_c^{(1)} = -p_c
\]

(30)

\[
(p_c^{(2)})^3 - (p_c^{(2)})^2 - p_c^{(2)} = -p_c
\]

(31)

\[
p_c^{(3)} = p_c
\]

(32)
Table 2. The three percolation concentrations \( p_c^{(2)} \), \( p_c^{(2)} \), and \( p_c^{(3)} \) for model II in both the annealed and quenched versions on various lattices. These values are all exact except for those of the simple cubic lattice which are based on good numerical estimates (Essam 1972).

<table>
<thead>
<tr>
<th>Lattice</th>
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<th>( p_c^{(2)} )</th>
<th>( p_c^{(3)} )</th>
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<td>0.4030</td>
</tr>
<tr>
<td>Triangular net</td>
<td>0.5935</td>
<td>0.5893</td>
<td>0.2919</td>
</tr>
<tr>
<td>Simple cubic</td>
<td>0.494</td>
<td>0.497</td>
<td>0.209</td>
</tr>
</tbody>
</table>

where \( p_c \) is the percolation concentration for the quenched bond problem on the skeleton lattice. A comparison of these concentrations for various lattices is given in table 2.

Model II, with three separate percolation concentrations is the one that would correspond more closely to real systems (supposing that \( I/J \) could be varied experimentally). Of course it is much easier to study the site problem experimentally, and all reasonable models would seem to lead to three percolation concentrations rather than two (e.g. Mn1₋₋₋Zn2₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋₋プラス;

We note that equations (23), (24), (30), (31), (32) for quenched systems all involve \( p_c \), the bond percolation concentration for the nearest-neighbour bond problem defined on the skeleton lattice. By replacing this by the value obtained in the annealed problem \( \frac{1}{3}[1 + \epsilon(K_s)][1 - \exp(-2K_s)] \) (Thorpe and Beeman 1976) we obtain the results for the annealed systems, (17), (18), (27), (28) and (29).

4. Conclusions

We have set up and studied the behaviour of a random Ising model with first- and second-neighbour interactions. We have obtained the general solution to the annealed version of this problem and studied the dilute ferromagnet in some detail. It has been shown that the annealed and quenched versions of this problem give very similar phase diagrams. In the case when \( I/J \ll 1 \), the phase diagram develops a 'tail' as we would expect intuitively. There is a limited amount of experimental evidence, but this does seem to occur (e.g. MnF2 see Baker et al 1961).

Acknowledgments

We would like to thank the referee for bringing the work of T Idogaki and N Uryu (J. Phys. Soc. Japan 43 845 1977) to our attention. These authors obtain phase boundaries for the site-diluted two-dimensional square net with second-neighbour interactions by using high-temperature expansions. They obtain the same kind of phase diagrams as in this paper (compare for example our figure 3 with their figure 8). This reinforces our belief that the general shape of the phase boundaries found in this paper are rather general and not dependent on using a particular model.
References

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