Elastic properties of rigid fiber-reinforced composites

J. Chen and M. F. Thorpe
Department of Physics and Astronomy, and Center for Fundamental Materials Research, Michigan State University, East Lansing, Michigan 48824
L. C. Davis
Research Laboratory, Ford Motor Company, Dearborn, Michigan 48121
(Received 30 August 1994; accepted for publication 17 January 1995)
We study the elastic properties of rigid fiber-reinforced composites with perfect bonding between fibers and matrix, and also with sliding boundary conditions. In the dilute region, there exists an exact analytical solution. Around the rigidity threshold we find the elastic moduli and Poisson's ratio by decomposing the deformation into a compression mode and a rotation mode. For perfect bonding, both modes are important, whereas only the compression mode is operative for sliding boundary conditions. We employ the digital-image-based method and a finite element analysis to perform computer simulations which confirm our analytical predictions. © 1995 American Institute of Physics.

I. INTRODUCTION
It is well known that fiber-reinforced composites can have more desirable properties than conventional homogeneous engineering materials, such as steel, aluminum, glass, and carbon. Using composites to replace single-component materials has become important in the polymer industry. Recently, researchers have done extensive finite element analyses to understand toughening and crazing in composites. This standard technique is powerful when only a small number of fibers or particles are embedded in a matrix. It is beyond current computational power to study the case in which hundreds of fibers or particles are randomly embedded in a matrix, which is a case that occurs in practical applications. In this article, we describe a digital-image-based model which can easily handle such random systems.

For practical fiber-reinforced composites, such as graphite fiber materials for tennis rackets and glass-epoxy composites, the fibers are much stiffer than the matrix. For example, glass is usually considered to be an isotropic material that has a Young's modulus of 76 GPa for E Glass. Compared with the Young's modulus of IMHS Epoxy, which is about 3.5 GPa, glass is about 20 times stiffer. Therefore, in glass-epoxy composites, we can regard the glass fibers as perfectly rigid inclusions to a reasonable first approximation. This assumption not only simplifies the mathematical calculations considerably, but also correctly gives the characteristic behavior of the composite and leads to some insights. Since this assumption of perfectly rigid fibers greatly simplifies the problem, it is adopted here. We study two interface conditions. In the first, the fibers are assumed to adhere perfectly to the matrix. In other words, we assume the boundary condition on the fiber-matrix interface to be perfect bonding. In the second case, we assume sliding boundary conditions, that is, the matrix on the surface of the fibers can move freely while remaining in contact (no delamination).

This article is organized as follows. In Sec. II, we present the exact elastic theory for random composites in the dilute limit and also discuss the modes existing in a random composite near the rigidity threshold. In Sec. III, we discuss the critical behavior of composites near the rigidity threshold and give the elastic constants for three different ordered structures. In Sec. IV, we give an improved approximation for the area bulk modulus near the rigidity threshold. In Sec. V, we discuss two different effective medium theories. In Sec. VI, we use the digital-image-based method to perform the simulation for perfect bonding boundary conditions. In Sec. VII, we use finite element analysis to study the case where the boundary conditions are pure sliding. Finally, a summary is given in Sec. VIII.

II. THEORY
A. Elasticity theory for a two-dimensional isotropic system
For a two-dimensional isotropic system, there are only two independent elastic constants. Once any two of these elastic constants are determined, all the other elastic coefficients are known. Suppose we choose (arbitrarily) the Young's modulus E and the area Poisson's ratio ν as the two basic elastic constants, then the area bulk modulus K, the shear modulus μ, and the longitudinal elastic stiffness constant C_{11} are obtained through the following equations:

$$K = \frac{E}{2(1-\nu)}, \quad \mu = \frac{E}{2(1+\nu)}, \quad C_{11} = \frac{E}{(1+\nu)(1-\nu)}.$$ (1)

B. Dilute limit
When the area fraction 1-p of rigid fibers is very small, the elastic moduli of composites can be obtained exactly. For simplicity, let the fiber-matrix interface be described as perfect bonding, which means that the fibers are rigidly attached to the matrix and no sliding on the interfaces is allowed. Under these conditions, the effective elastic properties of composites, such as the area bulk modulus K, shear modulus μ, Young's modulus E, and area Poisson's ratio ν, can be obtained analytically through the following equations:

---

Electronic mail: thorpe@pa.msu.edu

---
\[
K_0 = \frac{1-(1-p)\frac{1}{2+\nu_0}},
\]
\[
\mu_0 = \frac{1-(1-p)\frac{4}{3-\nu_0}},
\]
\[
\frac{E_0}{E} = \frac{1-(1-p)\left\{\frac{\nu_0}{1+\nu_0} + \frac{2(1+\nu_0)}{3-\nu_0}\right\}},
\]
\[
\nu = \nu_0 + (1-p)\frac{(1-\nu_0)(1-3\nu_0)}{3-\nu_0},
\]
where \(p\) is the area fraction of the matrix and the subscript 0 denotes matrix quantities. These results are only valid when \(p \to 1\), that is when the area fraction of fibers is very small. It is obviously true that the effective elastic moduli always increase when rigid fibers are added to the matrix. But whether the area Poisson’s ratio of the composite increases or decreases is more subtle and depends upon the area Poisson’s ratio of the matrix. If \(\nu_0 > 1/3\), after adding some rigid inclusions to the matrix, the area Poisson’s ratio of composite decreases; otherwise it increases.

C. Results near rigidity threshold

When the area fraction of fibers increases, the fibers touch each other, continuously across the sample; this occurs at a critical value \(p = p_c\). At this point, which is called the rigidity threshold, all of the elastic moduli go to infinity. For different geometrical arrangements of the fibers, \(p_c\) is different. When two rigid fibers are very close, the elastic energy is confined to the narrow region between them, which we define as the neck.

In Figs. 1(a) and 1(b), we show the general case in which two rigid fibers are embedded in a matrix. In this configuration, there exist only two relevant deformation modes. One is the compression mode which is shown in Fig. 1(a), and the other one is the rotation mode, which is shown in Fig. 1(b).

D. Compression mode

In compression mode, the external force \(F\) acts on the fibers and pushes them together. The force \(F\) can be expressed as

\[
F = A[Au_1 + Au_2] - A\Delta u,
\]
where \(\Delta u\) is the relative motion of the centers of two fibers under compression. \(A\) is an effective force constant which describes the relation between the applied force \(F\) and the relative motion of the fibers, and can be obtained by doing an appropriate integration over the neck region. In Fig. 1(a), the \(x\) axis is drawn through the center of the neck, with the origin at the narrowest part. To leading order, the width of the neck \(W(x)\), at a distance \(x\) from the center, is obtained by the parabolic approximation

\[
W(x) = w + \frac{x^2}{R},
\]
where \(w\) is the narrowest width of the neck and \(R\) is the radius of the two rigid fibers. Define \(R_e\) as the radius of the fibers when they just touch, then we have

\[
1 - p e \cdot R^2, \quad 1 - p e \cdot R_e^2.
\]
Since

\[
R_e = R + \frac{w}{2},
\]
we have

\[
\frac{w}{R} = \frac{p - p_e}{1 - p_e}
\]
close to percolation for any regular geometry. When a small strip of the neck at a distance \(x\), with width \(dx\) and length \(W(x)\), is under compression, the stress-strain relation is given by

\[
\frac{dF}{dx} = C_{11} \frac{\Delta u}{W(x)}.
\]
Integrating Eq. (8), we get

\[
F = \int_{-\infty}^{+\infty} C_{11} \frac{\Delta u}{w + (x^2/R)} \, dx,
\]
which gives the force constant \(A\) by using Eqs. (7) and (3)

\[
A = \pi C_{11} \frac{R}{w} \left(\frac{1 - p_e}{p - p_e}\right)^{1/2}.
\]
E. Rotation mode

The force constant $B$ for the rotation mode is obtained in a similar way. When the neck is acted on by a shear force $F$, as shown in Fig. 1(b), the two rigid fibers have a relative rotation $\Delta \phi = |\phi_1 - \phi_2|$. Therefore,

$$\frac{dF}{dx} = \mu_0 \frac{R \Delta \phi}{W(x)},$$

(11)

where $\mu_0$ is the shear modulus of the matrix. Defining $R = BR[\Delta \phi]$, and integrating Eq. (11) from $-\infty$ to $+\infty$, we obtain

$$B = \pi \mu_0 \sqrt{\frac{R}{w}} = \pi \mu_0 \left(\frac{1 - \nu_c}{p - \nu_c}\right)^{1/2}.$$  

(12)

III. CRITICAL BEHAVIOR

In this section, we study the asymptotic formulas for the effective elastic moduli of the composites when $p \rightarrow p_c$. Three ordered structures are studied, which are rigid fibers located on a triangular superlattice, a hexagonal superlattice, and a Kagome superlattice, respectively. These three lattices were chosen as they are elastically isotropic. For each structure, we give the asymptotic formulas for the effective Young’s modulus $E$, area bulk modulus $K$, shear modulus $\mu$, and the area Poisson’s ratio $\nu$ near the rigidity threshold.

A. Triangular arrangement

In Fig. 2, we show a composite with fibers on a triangular superlattice under the external macroscopic strain $\varepsilon_x$ in $x$ direction produced by the force $F$. No constraints are imposed on the material in the $y$ direction, other than the periodic boundary condition. Therefore, by calculating the energy per unit area, under an appropriate macroscopic strain, we can find the effective Young’s modulus $E$ and area Poisson’s ratio $\nu$. After that, all of the elastic moduli, such as the area bulk modulus $K$ and shear modulus $\mu$, are obtained by using Eqs. (1).

Let $\varepsilon_x$ be the longitudinal strain applied to the composite in $x$ direction, and $\varepsilon_y$ be the corresponding transverse strain in $y$ direction. Then we have

$$\varepsilon_x = \frac{2u}{a}, \quad \varepsilon_y = \frac{-2v}{\sqrt{3}a},$$

(13)

where $a$ is the interfiber distance between the centers of adjacent fibers and $u$, $v$ are the displacements of the fibers shown in Fig. 2. Therefore, $U$, the energy per cell (shown by the dashed lines in Fig. 2) is obtained by calculating the energy stored in the corresponding necks. We obtain

$$U = \frac{1}{2} A (2u)^2 + 2 \left[ \frac{1}{2} A (u \sin \phi_0 - v \cos \phi_0)^2 + \frac{1}{2} B (u \cos \phi_0 + v \sin \phi_0)^2 \right]$$

(14)

with $\phi_0 = \pi/6$. The area of the unit cell is

$$S_{cell} = \frac{\sqrt{3}}{2} a^2.$$  

(15)

Thus, the energy per unit area is given by

$$\frac{U}{S_{cell}} = \frac{2}{\sqrt{3} a^2} \left( 2 A u^2 + \frac{1}{4} A (u - \sqrt{3} v)^2 + \frac{1}{4} B (\sqrt{3} u + v)^2 \right).$$

(16)

Minimizing Eq. (16) with respect to $u$ (since there is no applied stress in the $y$ direction), we get

$$\frac{v}{u} = \frac{\sqrt{3} (1 - B/A)}{3 + B/A}.$$  

(17)

The area Poisson’s ratio $\nu$ is obtained by using Eqs. (13) and (17)

$$\nu = \frac{\varepsilon_y}{\varepsilon_x} = \frac{1 - B/A}{3 + B/A}.$$  

(18)

By definition

$$\frac{U}{S_{cell}} = \frac{1}{2} E \varepsilon_x^2,$$

(19)

so the effective Young’s modulus is given by

$$E = 2 \sqrt{3} A \left( \frac{1 + B/A}{3 + B/A} \right).$$

(20)

The effective area bulk modulus and the shear modulus are given by using Eqs. (1) as

$$K = \frac{\sqrt{3}}{2} A, \quad \mu = \frac{\sqrt{3}}{4} (A + B)$$

(21)

and $K$ relates only to the compression mode as expected. From Eqs. (1), (10), and (12), we find

$$A = \frac{C_{11}^{0}}{\mu_0} = \frac{2}{1 - \nu_0},$$

(22)

where $\nu_0$ is the area Poisson’s ratio of the matrix. Therefore, by substituting Eq. (22) into Eqs. (18), (20), (21), and using Eqs. (10), (12), we get

$$\nu = \frac{1 + \nu_0}{7 - \nu_0},$$

(23)
Thus, the energy per unit area is given by
\[ U = \frac{4}{3\sqrt{3}a^2} \left[ A\nu^2 + \frac{1}{8} A(\sqrt{3}\nu + v - w)^2 + \frac{1}{8} B(u - \sqrt{3}v + \sqrt{3}w)^2 \right] . \] (31)

If we minimize Eq. (31) with respect to \( u, w \), respectively, and solve those equations, to find that
\( u = 0 \),
and
\[ w = \frac{\sqrt{3}(1 - B/A)}{1 + 3B/A} . \] (32)

Using Eqs. (29), (32) and the definition of the area Poisson's ratio \( \nu = \varepsilon_{T}/\varepsilon_L \), we obtain
\[ \nu = \frac{1 - B/A}{1 + 3B/A} . \] (33)

From Eqs. (1), (19), (29), (31), and (32), we get
\[ E = \frac{4B}{\sqrt{3} \left( 1 + 3B/A \right)} , \quad K = \frac{A}{2\sqrt{3}} , \quad \mu = \frac{1}{\sqrt{3}} \frac{B}{1 + B/A} . \] (34)

Thus, from Eqs. (10), (12), and (22), we get
\[ \nu = \frac{1 + \nu_0}{5 - 3\nu_0} . \] (35)

\[ \frac{\nu_0}{\sqrt{3}} \left( \frac{1 + \nu_0}{1 - \nu_0} \right)^{1/2} \] (36)
\[ \mu = \frac{\sqrt{3}}{2\pi} \left( 3 - v_0 \right) \left( \frac{p_c - 1}{1 - p_c} \right) ^{1/2} \] (37)
\[ E_0 = \frac{\sqrt{3}}{4\pi} \left( 1 + \nu_0 \right) \left( 5 - 3\nu_0 \right) \left( \frac{p_c - 1}{1 - p_c} \right)^{1/2} \] (38)
where \( p_c \) is given by
\[ p_c = 1 - \frac{\pi}{2\sqrt{3}} = 0.0931 . \] (27)

B. Hexagonal arrangement

Another ordered structure consists of rigid fibers located on a hexagonal superlattice (Fig. 3). The effective elastic properties are obtained by using the same technique as we used in the previous subsection. Defining \( \phi_0 = \pi/6 \), then the total energy per cell (shown by the dashed lines in Fig. 3) is given by
\[ U = \frac{4}{3\sqrt{3}a^2} \left[ \right] \] (29)
where \( u, v, w \) are defined in Fig. 3 and obey the following relations:
\[ \varepsilon_L = \frac{2u}{\sqrt{3}a} , \quad \varepsilon_T = \frac{2w + 2v}{3a} . \] (29)

It is necessary to define a third distance variable \( w \), because of the two inequivalent types of fibers in the infinite matrix. The area of this cell is
\[ S_{cell} = \frac{3\sqrt{3}}{2} a^2 . \] (30)

C. Kagome arrangement

Another interesting case is the Kagome superlattice (Fig. 4). By defining \( \phi_0 = \pi/6 \) and assuming that the interfiber distance in \( 3a \), we find that the total energy per unit cell \( U \) can be written as
From Eqs. (1), the area bulk modulus and the shear modulus are

\[ K = \frac{\sqrt{3}}{4} A, \quad \mu = \frac{\sqrt{3}}{8} A. \]  

Combining Eqs. (10), (46), and (47), we obtain the elastic moduli for the Kagome arrangement

\[ K_0 = \frac{2}{\sqrt{3} \pi} \left( 1 + \nu_0 \right) \left( \frac{p - p_c}{1 - p_c} \right)^{1/2}, \]

\[ \mu_0 = \frac{4}{\sqrt{3} \pi} \left( 1 - \nu_0 \right) \left( \frac{p - p_c}{1 - p_c} \right)^{1/2}, \]

\[ E_0 = \frac{\sqrt{3}}{\pi} \left( 1 + \nu_0 \right) \left( 1 - \nu_0 \right) \left( \frac{p - p_c}{1 - p_c} \right)^{1/2}, \]

where \( p_c \) is given by

\[ p_c = 1 - \frac{\sqrt{3}}{8} = 0.3198. \]  

### IV. The Area Bulk Modulus Near the Rigidity Threshold

In Eqs. (24), (36), and (48), we have expressions for the area bulk modulus to leading order as the rigidity threshold is approached. These formulas can be improved somewhat by including the next order term enabling us to describe the critical behavior over a larger range of \( p \). This is most straightforward for the area bulk modulus because of the high symmetry associated with the distortions.

In Fig. 5(a), we show three rigid fibers with radius \( R \) located on a triangular superlattice. This time we use the full expression for \( W(x) \) rather than the parabolic approximation. We redefine \( W \) as a function of \( \theta \) by using \( x = R \sin \theta \), where \( \theta \) is the angle relative to the line joining the centers of two nearest fibers

\[ W(\theta) = w + 2R \left( \cos \theta + 2\cos \theta \sin \theta \right). \]

Defining \( \dot{\theta} = \pi/6 \) and \( b = W(\dot{\theta}) \), we can decompose Fig. 5(a) into three necks (\( \theta \) from 0 to \( \dot{\theta} \)) plus an equilateral triangular matrix with side length \( b \). Suppose this composite is under a hydrostatic pressure which induces a uniform radial strain \( \epsilon \), then the force constant for the compression mode is obtained through

\[ F = C_{11}^0 \Delta u \int_{-\theta}^{\theta} d(\theta) W(\theta). \]

Doing this integral and using Eq. (52), we get the (improved) force constant for the compression mode

\[ A = C_{11}^0 \left[ \frac{\left( \frac{1 - \nu_0}{p - p_c} \right)^{1/2} - \frac{\phi_0}{\pi} \frac{1}{\pi \tan(\phi_0/2)} - \frac{1}{3} \right] \epsilon. \]

where only the first term (that diverges as the rigidity threshold is approached) was obtained previously. The total energy is the energy stored in the necks plus the energy stored in the
central equilateral triangular piece of matrix. Noticing that only 3/2 necks are involved inside the equilateral triangle with side length $a$, we get

$$3 \frac{1}{2} A(ea)^2 + \frac{\sqrt{3}}{4} b^2 2K_0 e_0^2 = \frac{\sqrt{3}}{4} a^2 2K_0 \varepsilon^2.$$  \hspace{1cm} (55)

where $e_0$ is the (assumed) uniform strain in the inner equilateral triangular matrix. Since

$$\varepsilon = \frac{\Delta u}{a}, \quad e_0 = \frac{\Delta u}{b},$$

we have

$$e_0 = \frac{a}{\varepsilon},$$ \hspace{1cm} (57)

Combining Eqs. (55) and (57), we get

$$\frac{K_0}{K} = \frac{1}{\alpha} \left( \frac{p-p_c}{1-p_c} \right)^{1/2} \left[ 1 - \frac{\beta}{\alpha} \left( \frac{p-p_c}{1-p_c} \right)^{1/2} \right],$$ \hspace{1cm} (58)

where

$$\alpha = \frac{\sqrt{3} \pi}{1 + \nu_0}, \quad \beta = 1 - \alpha \left( \frac{1}{\beta} + \frac{1}{\pi \tan (\pi/6)} \right).$$ \hspace{1cm} (59)

The modified formulas for hexagonal and Kagome arrangements are obtained in a similar way, using Fig. 5(b) and 5(c). (The detailed calculations are provided in the Appendix). For the hexagonal arrangement, shown in Fig. 5(b), we find

$$\frac{K_0}{K} = \frac{1}{\alpha} \left( \frac{p-p_c}{1-p_c} \right)^{1/2} \left[ 1 - \frac{\beta}{\alpha} \left( \frac{p-p_c}{1-p_c} \right)^{1/2} \right],$$ \hspace{1cm} (60)

where

$$\alpha = \frac{\pi}{\sqrt{3}(1 + \nu_0)},$$

and

$$\beta = 1 - \alpha \left( \frac{1}{3} + \frac{1}{\pi \tan (\pi/6)} \right).$$ \hspace{1cm} (61)

For the Kagome arrangement, shown in Fig. 5(c), its modified area bulk modulus is

$$\frac{K_0}{K} = \frac{1}{\alpha} \left( \frac{1-p_c}{1-p_c} \right)^{1/2} \left[ 1 - \frac{\beta}{\alpha} \left( \frac{1-p_c}{1-p_c} \right)^{1/2} \right],$$ \hspace{1cm} (62)

where

$$\alpha = \frac{\sqrt{3} \pi}{1 + \nu_0}, \quad \beta = 1 - \alpha \left( \frac{1}{\beta} + \frac{1}{\pi \tan (\pi/2)} \right).$$
\[ \alpha = \frac{\sqrt{3} \pi}{2(1 + v_0)}, \]
and
\[ \beta = 1 - \frac{\sqrt{3} \pi}{8(1 + v_0)}, \]  
(63)

and Eqs. (58), (60), (62) are the same as Eqs. (24), (36), (48), except that the next higher order term in \((p - p_1)/(1 - p_2)\) has been included approximately.

V. EFFECTIVE MEDIUM THEORY

For a composite with randomly distributed fibers embedded in a matrix, we can use effective medium theory to calculate its effective elastic properties approximately, away from the dilute limit. One such effective medium theory is the self-consistent method, and another one is the three-phase model which is perhaps a somewhat more sophisticated approach. In the self-consistent method, the elastic properties are obtained by solving the following equations,\(^8\)

\[
\frac{1 + v_0}{1 - v_0} = \frac{1}{1 - \nu} \left[ \frac{1 - 2(1 - p)/(1 + \nu)}{1 - 4(1 - p)/(3 - \nu)} \right],
\]

\[
\frac{K_0}{K} = 1 - (1 - p) \frac{2}{1 + \nu},
\]

\[
\frac{\mu_0}{\mu} = 1 - (1 - p) \frac{4}{3 - \nu},
\]

\[
\frac{4}{E} = \frac{1}{K} + \frac{1}{\mu}.
\]

In the three-phase model, the area bulk modulus is given by\(^9\)

\[
\frac{K_0}{K} = \frac{p}{1 + (1 - p)(1 - v_0/1 + v_0)}.
\]

The shear modulus \(\mu\) is found in a similar but more complicated way,\(^9\) and hence using Eq. (1) all the elastic moduli are known.

VI. DIGITAL-IMAGE-BASED METHOD

In this section, we use a spring-grid scheme to calculate the transverse effective elastic moduli of fiber-reinforced materials.\(^4\) The idea behind this method is to use a fixed triangular spring network with central forces to describe the composite. By tuning the spring constants, the triangular network can be made to have the same elastic properties as the matrix. A cartoon of such a spring network, with spring constants \(\alpha\), \(\beta\), and \(\gamma\), is shown in Fig. 6. From our previous work,\(^4\) we know that such an arrangement makes the system isotropic with the area bulk modulus \(K\) and transverse shear modulus \(\mu\) given by

\[
K = \frac{1}{\sqrt{12}} (\alpha + \beta + \gamma),
\]

(64)

\[
\mu = \sqrt{\frac{27}{16}} \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right)^{-1}.
\]

Once \(K\) and \(\mu\) are determined, we can obtain the Young’s modulus \(E\) and area Poisson’s ratio \(\nu\) through Eqs. (1)

\[
E = \frac{2\sqrt{3}(\alpha + \beta + \gamma)}{3 \left[ 1 + \frac{2}{9} (\alpha + \beta + \gamma) \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right) \right]},
\]

(65)

\[
\nu = 1 - \frac{2}{1 + \frac{2}{9} (\alpha + \beta + \gamma) \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right)}.
\]

In our simulation, we use a 210\(\times\)210 triangular lattice as a unit supercell and impose periodic boundary conditions. We use two different sets of spring constants \(\alpha\), \(\beta\), and \(\gamma\). One set is chosen to give the elastic constants of the continuum matrix, in which \(\alpha, \beta, \gamma\) are all of order 1. The other set is chosen to be \(\alpha = \beta = \gamma = 1000\), which simulates the rigid fibers. We find that a 1000:1 ratio is rigid enough to give sufficiently accurate results.

After setting up the grid, we impose an external uniform strain in the \(x\) direction, \(e_x = 0.001\), on the unit cell and let the system relax with no constrains in the \(y\) direction. We use the conjugate gradient method\(^10\) to obtain the stresses. By calculating the energy per unit area and by measuring the change of the length of the cell in the \(y\) direction, we find Young’s modulus \(E\) and area Poisson’s ratio \(\nu\). After that, all the other elastic constants can be found by using Eqs. (1).

For ordered structures, we did the simulation on the three different arrangements of rigid fibers: triangular, hexagonal, and Kagome. We show the flow diagrams for the area Poisson’s ratios in Figs. 7(a)–7(c). We plot the analytical results in the dilute region by using Eqs. (2), as shown by solid lines in Figs. 7(a)–7(c). The interesting point is that in general there exists no fixed points in these diagrams. This is quite different from the results we obtained for the case when the inclusions were holes rather than rigid fibers.\(^4\) In porous composites, the fixed point always exists in the flow diagram for the area Poisson’s ratio, and its value at percolation is independent of the size and the shape of the holes. This result has been verified by computer simulations\(^4\) and has been proven analytically.\(^11–13\) This is true because there exists only one independent relevant elastic deformation in porous composites near percolation. Therefore, all the elastic constants are scaled by this unique parameter near the percola-
tion threshold, which makes the area Poisson's ratio flow to a constant. But this argument is not true in fiber-reinforced composites, because in this case there are two independent deformation modes: compression and rotation. As we have seen from Eqs. (10) and (12), the compression mode couples to the elastic constant $C_{11}$ of the matrix and the rotation mode couples to the shear modulus $\mu$ of the matrix. Therefore, in general, the elastic moduli of fiber-reinforced composites with perfect bonding at the interface depend upon two independent parameters of the matrix rather than one. There is no fixed point for the area Poisson's ratio in fiber-reinforced composites in general. We have also studied the same structures recently by using finite element analysis with the commercial software ABAQUS \textsuperscript{14} and obtained similar results on samples containing a smaller number of composites.

However under certain arrangements of rigid fibers, for example, the Kagome lattice, a fixed point does exist in the area Poisson's ratio flow diagram, because of the special symmetry of the arrangement. This result has been proven analytically in Eq. (45) and has been verified by our simulation results [Fig. 7(c)]. Therefore, our conclusion is that the fixed point in the area Poisson's ratio flow diagram does not exist in general for fiber-reinforced composites, but it can exist in certain cases with special symmetry. If the boundary condition is pure sliding on the fiber-matrix interface, which means that there is no friction force on interfaces, then there will be no rotation mode in the energy expression. Therefore, all of the elastic moduli scale with the single deformation mode, the neck compression, which always gives a fixed point in the area Poisson's ratio flow diagram. (See Sec. VII.)

A big advantage of the digital-image-based method used here is that it is easy to study the case in which hundreds of fibers are embedded in matrix, whereas finite element modeling has no such ability at present.\textsuperscript{15} For the overlapping fiber case, a configuration is generated by randomly placing

![FIG. 7. Showing the area Poisson's ratio flow diagram for different arrangements of rigid fibers (perfect bonding) and for different matrix Poisson's ratio \(\nu_m\). (a) Triangular arrangement; (b) hexagonal arrangement; (c) Kagome arrangement; (d) random arrangements (the symbol $\circ$ stands for the overlapping case and $\bullet$ for the nonoverlapping case).](image1)

![FIG. 8. The area bulk modulus as a function of $p$ for different arrangements of rigid fibers and for a matrix area Poisson's ratio of 1/3, for (a) triangular arrangement; (b) hexagonal arrangement; (c) Kagome arrangement. The solid lines in the dilute limit when $p$ is close to 1 are the exact dilute results. The dashed lines near the rigidity threshold are the asymptotic square root form and the solid lines are the improved approximation described in the text. In panel (d) for random arrangements, the symbol $\circ$ stands for the overlapping case and $\bullet$ for the nonoverlapping case. The solid line is the self-consistent theory and the dashed line is the three-phase theory. The dashed line in (d) is also the Hashin's lower bound for the area bulk modulus (Ref. 17).](image2)
fibers on the matrix. The center of the rigid fibers are determined by a random number generator. For the nonoverlapping case, we use the Metropolis version of the Monte Carlo method to generate configurations.\textsuperscript{16} The unit cell is still 210 ×210 and the diameter of the rigid fibers is chosen to be 11. For this choice of the diameter, there are enough pixels inside a rigid fiber, to make the surfaces of the fibers smooth after digitalizing.

In Fig. 7(d), we show the area Poisson's ratio flow diagram for both overlapping and nonoverlapping cases. In both cases, we did an average over six configurations and the error bars from this averaging are shown in Fig. 7(d). The solid lines are the results obtained from the self-consistent method\textsuperscript{9} and the dashed lines are the results of the three-phase model.\textsuperscript{9} As we see from Fig. 7(d), the three-phase model is better than the self-consistent method, in the sense that it agrees with the simulation data that no fixed point exists in the area Poisson's ratio flow diagram for random fiber-reinforced composites. It is also true that the data for the nonoverlapping case is closer to the prediction of the three-phase model, and the data for the overlapping case is closer to the prediction of the self-consistent method.

In Fig. 8 we have plotted the area bulk modulus with respect to the area fraction of matrix for all four geometries. The area Poisson's ratio of the matrix is \( v_0 = \frac{1}{3} \). In Figs. 8(a)–8(c), the solid lines around \( p = 1 \) are the exact results in the dilute region, which are given by Eqs. (2); the solid lines around the rigidity threshold \( p_C \) are the improved area bulk modulus obtained by Eqs. (58), (60), and (62); the dashed lines around \( p_C \) are results given by Eqs. (24), (36), and (48), which are to lowest order. Around the rigidity threshold, the analytical results can explain the simulation data up to \( p - p_C = 0.2 \). In Fig. 8(d), the solid lines are the analytical results obtained from the self-consistent method by using Eqs. (64), and the dashed line is the result obtained from the three-phase model. The latter corresponds to Hashin's lower bound\textsuperscript{13} on the area bulk modulus (upper bound on \( K_0/K \)) of a composite with rigid inclusions. The upper bound on \( K \) is undefined in this case. The sharper third-order lower bounds\textsuperscript{18} for random overlapping and nonoverlapping identical, rigid cylinders differ only slightly from the Hashin bound. In the dilute limit, simulation data agree nicely with analytical results until \( p = 0.8 \). Again, it is still true that the three-phase model gives better results for the nonoverlapping case, and the self-consistent method agrees more closely to the overlapping case. A mean-field theory for \( K \) valid near \( p_C \) has been given in Ref. 19.

### VII. PURE SLIDING BOUNDARY CONDITION

#### A. Dilute results

The other (tractable) boundary condition between the rigid fibers and the matrix is pure sliding without friction in which no delamination is allowed. This means that the surfaces of the fiber and matrix always remain in contact. In the dilute limit, the effective elastic moduli can be obtained through the elastic strain energy. By assuming the pure sliding boundary condition, the effective area bulk modulus \( K \), shear modulus \( \mu \), Young's modulus \( E \), and the area Poisson's ratio \( \nu \) for 2D composites are given by\textsuperscript{5}

\[
\begin{align*}
K_0 - K = 1 - (1 - p) \frac{2}{1 + v_0}, \\
\mu_0 - \mu = 1 - (1 - p) \frac{4}{5 - v_0}, \\
E_0 - F = 1 - (1 - p) \frac{1 - v_0}{1 + v_0 + \frac{2(1 + v_0)}{5 - v_0}} , \\
\nu = v_0 + (1 - p) \frac{3(1 - v_0)^2}{5 - v_0},
\end{align*}
\]

where \( p \) is the area fraction of the matrix, and the subscript 0 denotes the corresponding quantities for the matrix. We note that the area Poisson's ratio always increases when a small amount of rigid inclusions with pure sliding surface are added to the matrix. This is different from the case with perfectly bonding boundary conditions. In that case, the area Poisson's ratio increases if the area Poisson's ratio of matrix is less than \( 1/3 \) and decreases otherwise.\textsuperscript{4} These dilute results provide a good check of the correctness of simulation results.

#### B. Rigidity threshold

If there is pure sliding at the fiber interface, then all of our discussion in Sec. III is valid except that \( R \), the effective force constant for rotation mode, must be set equal to zero, as there is no restoring force for this kind of motion. In two dimension, since it is pure sliding on the inclusion-matrix interface, there exists no rotation mode associated with the elastic energy. Therefore, near the rigidity threshold, all of the elastic moduli scale to the single deformation mode that is proportional to \( A \) and hence to \( C_{11}^0 \). This means that there always exists a fixed point in the flow diagram for the area Poisson's ratio.

Setting \( B = 0 \) and following the procedures in Sec. III, we find that the fixed points for triangular, hexagonal, and Kagome superlattices are \( v_r = 1/3, v_c = 1, \) and \( v_c = 1/3, \) respectively.

#### C. Finite element analysis

For rigid inclusions with pure sliding boundary conditions, the spring-grid scheme used before is very difficult to implement. This is because slip rings and the like would be needed at the interface within this scheme. So we switched to Abaqus, a commercial package for finite element analysis.\textsuperscript{14}

To discretize the matrix, the four-node bilinear plain strain elements (CPE4) are used for 2D composites.\textsuperscript{14} Since there is pure sliding on the interface between the rigid inclusions and the matrix, the nodes on this interface can move freely under the constraint that the distances between these nodes and the center of the inclusion are fixed. The same three ordered structures are studied, which are rigid inclusions located on a triangular superlattice, hexagonal superlattice, and Kagome superlattice. The total number of elements...
for each case is about 1000, which is large enough to obtain accurate results. We doubled the number of elements, but we found that the error due to this size effect was less than 0.5%. As before, periodic boundary conditions were used throughout.

In Fig. 9, we plot the area Poisson's ratio $\nu$ of composites against the area fraction of the matrix $p$ for different $\nu_0$, the area Poisson's ratio of the matrix. We find that the area Poisson's ratios do flow to the predicted fixed points which are marked by a star. There exists a very rapid change in the area Poisson's ratio when $p$ approaches the rigidity threshold $p_c$. We would have liked to set $p$ closer to $p_c$ with the numerical simulations, but were limited by the relatively small number of elements that could be handled by ABAQUS, on a workstation using a reasonable amount of CPU time.

VIII. SUMMARY

In this article, we studied the elastic properties of rigid fiber-reinforced composites. The interface between the fibers and the matrix was assumed to be either perfectly bonded or to allow pure sliding. We studied three ordered structures in which rigid fibers are located on triangular, hexagonal, and Kagome superlattices, and two random structures which are overlapping and nonoverlapping rigid fibers randomly embedded in the matrix. In the dilute limit, the elastic moduli and the area Poisson's ratio are known exactly. Around the rigidity threshold, we obtained these quantities approximately by calculating the energies in the narrow necks between adjacent rigid fibers. We noted that all of the effective elastic moduli for ordered structures $\sim (p-p_c)^{-1/2}$ when $p \to p_c$ for these two-dimensional geometries.

It is interesting to speculate what happens to rigid inclusions in higher dimensionalities. In three dimensions, the inclusions would be spheres, and the force between two adjacent spheres is given by

$$F = \int_0^{x_0} C_{11}^0 \Delta \nu \frac{2\pi \nu dx}{w + \frac{x}{R}}$$

rather than by Eq. (9). Here $x_0$ is some fairly large cutoff. It is not possible to extend the integral to infinity in this case as was done in two dimensions. The difference is that the integral now goes over a circular region centered on the neck axis. The force constant $A$ by using Eq. (7) is

$$A = \pi C_{11}^0 R \ln \left( \frac{R}{w} \right) \pi C_{11}^0 \ln \left( \frac{1-p_c}{p-p_c} \right).$$

This leads to the elastic moduli diverging logarithmically at the rigidity threshold as

$$\frac{K_0}{\mu} \sim \mu_0 \pi C_{11}^0 \ln \left( \frac{1-p_c}{p-p_c} \right)$$

rather than by Eq. (9). Here $x_0$ is some fairly large cutoff. It is not possible to extend the integral to infinity in this case as was done in two dimensions. The difference is that the integral now goes over a circular region centered on the neck axis. The force constant $A$ by using Eq. (7) is

$$A = \pi C_{11}^0 R \ln \left( \frac{R}{w} \right) \pi C_{11}^0 \ln \left( \frac{1-p_c}{p-p_c} \right).$$

This leads to the elastic moduli diverging logarithmically at the rigidity threshold as

$$\frac{K_0}{\mu} \sim \mu_0 \pi C_{11}^0 \ln \left( \frac{1-p_c}{p-p_c} \right)$$

rather than by Eq. (9). Here $x_0$ is some fairly large cutoff. It is not possible to extend the integral to infinity in this case as was done in two dimensions. The difference is that the integral now goes over a circular region centered on the neck axis. This is typical when the result is logarithmic behavior. Equation (69) has been verified by finite element analysis.20,21

Nevertheless the necks do act independently in the critical region in both two and three dimensions. In higher dimensions (four and above), so beloved by physicists interested in phase transitions, the energy is not so concentrated at the center of the neck as in two dimensions, as was shown by the necessity to use a cutoff in the integral. This is typical when the result is logarithmic behavior. Equation (69) has been verified by finite element analysis.20,21

We also developed the digital-image-based method to simulate ordered and random composites and obtained the effective elastic properties over the whole range of the area fraction of the matrix. We showed that the fixed point in the area Poisson's ratio flow diagram does not exist in general for perfect bonding, which is quite different from the results for porous composites.4 But the Kagome arrangement of rigid fibers, where we find a fixed point in the area Poisson's ratio flow diagram, is an exception, because of the special symmetry involved which eliminates the rotation mode. We also find that fixed points always exist if the boundary condition between fibers and matrix is pure sliding, which

![Figure 9](https://example.com/figure9.png)

**FIG. 9.** The area Poisson's ratio flow diagram for different arrangements of rigid fibers (pure sliding) and for different matrix Poisson's ratio $\nu_0$. (a) Triangular arrangement; (b) hexagonal arrangement; (c) Kagome arrangement. The (exact) calculated fixed point at the rigidity threshold is shown by a star in each case, and the exact results at small $(1-p)$ by the solid lines.
means that the portion of the matrix on the fiber-matrix interface can move freely on the surface of fibers.

ACKNOWLEDGMENTS

We would like to thank A. R. Day and I. M. Jasiuk for interesting discussions. This research is supported by the Composite Materials and Structures Center at Michigan State University.

APPENDIX: IMPROVEMENTS IN THE EFFECTIVE AREA BULK MODULUS

I. Hexagonal arrangement

Referring to Fig. 5(b) the fiber-fiber distance is

\[ a = w + 2R, \]  \hspace{1cm} (A1)

where \( w \) is the narrowest width of the neck and \( R \) is the radius of the fibers. The length of a side of the inner hexagonal matrix is

\[ b = W(\phi_0) - w + 2R(1 - \cos \phi_0) - w + R, \]  \hspace{1cm} (A2)

where \( \phi_0 = \pi/3 \). Under hydrostatic pressure, the centers of two nearest fibers are compressed by an amount \( \Delta u \). Thus, the force caused by this compression is given by

\[ F = C_{11}^0 \Delta u \int_{\phi_0}^{\phi_0 + \phi_0} d(R \sin \theta) W(\theta), \]  \hspace{1cm} (A3)

which gives the force constant through the definition \( F = A\Delta u \):

\[ A = C_{11}^0 \pi \left( \frac{1 - p_e}{p - p_e} \right)^{1/2} - \frac{\phi_0}{\pi} \frac{1}{\pi \tan(\phi_0/2)}. \]  \hspace{1cm} (A4)

The leading term in Eq. (A4) was found previously in the text by extending the integration in Eq. (A3) to infinity in both directions. The other two terms in Eq. (A4) represent corrections which we now keep.

Since the necks on the boundary of this cell are shared with its neighbors, there are only three full necks belonging to this cell. Also, by noticing the area of this cell is \( 6\sqrt{3}/4a^2 \), and the area of the inner hexagonal matrix is \( 6\sqrt{3}/2b^2 \), we can define the effective area bulk modulus \( K \) through the following equation:

\[ \frac{1}{2} A(ea)^2 + 6 \frac{\sqrt{3}}{4} b^2 2K_0 e_0^2 = 6 \frac{\sqrt{3}}{4} a^2 2K e^2. \]  \hspace{1cm} (A5)

where \( e \) is the uniform strain on the composite caused the hydrostatic pressure, and \( e_0 \) is the strain inside the inner hexagonal matrix. Since

\[ e = \frac{\Delta u}{a}, \quad e_0 = \frac{\Delta u}{b}, \]  \hspace{1cm} (A6)

we get

\[ e_0 = e \left( \frac{a}{b} \right). \]  \hspace{1cm} (A7)

Solving Eq. (A5) by using the relation (A7), we get

\[ K = \frac{1}{2\sqrt{3}} A + K_0. \]  \hspace{1cm} (A8)

Then, combining Eqs. (A4) and (A8), we have

\[ \frac{K}{K_0} = \frac{\pi}{\sqrt{3}(1 + \nu_0)} \left( \frac{\sqrt{R}}{w \pi} \frac{\phi_0}{\nu} \frac{1}{\pi \tan(\phi_0/2)} + 1 \right). \]  \hspace{1cm} (A9)

Keeping the leading two terms and using \( w/R = (p - p_e)/(1 - p_e) \), we have

\[ \frac{K}{K_0} = \alpha \left[ \beta \left( \frac{1 - p_e}{p - p_e} \right)^{1/2} \right], \]  \hspace{1cm} (A10)

where

\[ \alpha = \frac{\pi}{\sqrt{3}(1 + \nu_0)}, \]  \hspace{1cm} and

\[ \beta = 1 - \alpha \left( \frac{1}{3} + \frac{1}{\pi \tan(\pi/6)} \right). \]  \hspace{1cm} (A11)

II. Kagome arrangement

Referring to Fig. 5(c) for the Kagome lattice, the fiber-fiber distance is \( \sqrt{3}a \). So we have

\[ \sqrt{3}a = w + 2R. \]  \hspace{1cm} (A12)

Assume that the relative movement of nearest fibers is \( \Delta u \) under a hydrostatic pressure, which gives a macroscopic strain \( e = \Delta u/\sqrt{3}a \) on the composite. The force acting on a neck due to this hydrostatic pressure is

\[ F = C_{11}^0 \Delta u \int_{\phi_2}^{\phi_1 + \phi_2} d(R \sin \theta) W(\theta), \]  \hspace{1cm} (A13)

which gives the force constant through definition \( F = A\Delta u \)

\[ A = -\frac{1}{2} C_{11}^0 (\phi_1 + \phi_2) \]

\[ + C_{11}^0 \sqrt{\frac{\sqrt{1 + (w/2R)}}{w}} \left[ \arctan \left( \frac{2[\tan(\phi_2/2)]^{1/2} \sqrt{1 + (w/4R)}}{\sqrt{w/R}} \right) + \arctan \left( \frac{2[\tan(\phi_2/2)]^{1/2} \sqrt{1 + (w/4R)}}{\sqrt{w/R}} \right) \right]. \]  \hspace{1cm} (A14)
where $\phi_1 = \pi/3$ and $\phi_2 = \pi/6$.

Let the length of the side of the inner hexagonal matrix be $b$ and the strain inside it be $\varepsilon_1$. Let the length of the side of the corner equilateral triangular matrix be $c$ and the strain inside it be $\varepsilon_2$. By noticing that the fibers are perfectly rigid, we have

$$\varepsilon_1 = \frac{\Delta u}{b}, \quad \varepsilon_2 = \frac{\Delta u}{c}. \quad (A15)$$

Therefore,

$$\varepsilon_1 = \varepsilon \left( \frac{\sqrt{3}a}{b} \right),$$

and

$$\varepsilon_2 = \varepsilon \left( \frac{\sqrt{3}a}{c} \right). \quad (A16)$$

By calculating the total energy inside this primitive cell, as shown in Fig. 5(c), we have

$$6 \left[ \frac{1}{2} A \left[ \cos \phi_2 \right]^2 + 6 \frac{\sqrt{3}}{4} b^2 K_0 \varepsilon_1^2 + 2 \frac{\sqrt{3}}{4} c_2 2 K_0 \varepsilon_2^2 \right] = 6 \frac{\sqrt{3}}{4} (2a)^2 2 K_0 \varepsilon^2. \quad (A17)$$

Solving Eq. (A17) by using Eq. (A16), we get

$$K = \frac{\sqrt{3}}{4} A + K_0. \quad (A18)$$

From Eqs. (A14), (A18) and keeping only the first two leading terms, we obtain the effective area bulk modulus.

$$K_0 \frac{1}{K} = \frac{1}{\alpha} \left( \frac{1 - p_c}{1 - p_c} \right)^{1/2} \left[ 1 - \frac{\beta}{\alpha} \left( \frac{1}{1 - p_c} \right)^{1/2} \right], \quad (A19)$$

where

$$\alpha = \frac{\sqrt{3} \pi}{2(1 + \nu_0)}, \quad \beta = 1 - \frac{\sqrt{3} \pi}{8(1 + \nu_0)}, \quad (A20)$$

and

$$\rho_c = 1 - \frac{\sqrt{3} \pi}{8} \approx 0.3198. \quad (A21)$$