Elastic moduli of two dimensional materials with polygonal and elliptical holes

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We study the effective elastic moduli of two-dimensional composite materials containing polygonal holes. In the analysis we use a complex variable method of elasticity involving a conformal transformation. Then we take a far field result and derive the effective elastic constants of composites with a dilute concentration of polygonal holes. In the discussion we use the recently-stated Cherkaev-Lurie-Milton theorem, which gives general relations between the effective elastic constants of two-dimensional composites. We also discuss known results for elliptical holes in the context of the present work.

1. INTRODUCTION

We study the effective elastic moduli of two-dimensional (2D) composite materials with polygonal holes. First, we find the stress and displacement fields using a complex variable method of elasticity (Muskhelishvili, 1953) and the Schwarz-Christoffel conformal transformation. Then we derive the effective elastic constants of composites with a dilute concentration of polygonal holes by taking the far field result. In addition we discuss the results on elliptical holes.

This paper is an extension of our earlier paper (Jasiuk et al, 1992) in which we obtained the results for polygonal geometries numerically by using a finite difference approach. In this paper we use analytical methods and also address the anisotropy of the effective elastic constants. The approach followed here is similar to the one in Thorpe (1992) in the context of conductivity.

We have recently addressed the problem of a 2D material containing circular holes (Day et al, 1992; Thorpe and Jasiuk, 1992; Jun and Jasiuk, 1993) and obtained some interesting new results. These include the observation that the 2D elastic Young's modulus of a material containing holes is independent of the Poisson's ratio of the matrix and the 2D effective Poisson's ratio flows to a fixed point as the percolation threshold is reached. These results were initially obtained numerically (Day et al, 1992), but they are exact and can be proved by the recently stated Cherkaev-Lurie-Milton theorem (Cherkaev et al, 1992), which we refer to as the CLM theorem. This theorem applies to 2D linear elastic materials with general anisotropy and an arbitrary geometry. It reduces the parameter space of the local stress fields and the effective elastic properties of composite materials and its general result for stress fields agrees with the results of Michell (1899) and Dundurs (1967, 1969), as discussed by Thorpe and Jasiuk (1992). In this paper we explore further the applications of the CLM theorem for materials with holes.

Other related recent studies are due to Kachanov et al (1994), who studied materials with holes and cracks. Christensen (1993) who explored the extensions of the CLM theorem for 3D materials with holes, Zimmerman (1986) who considered the compressibility of 2D materials with holes of arbitrary shapes, and Movchan and Serekov (1992) and Movchan (1992) who used elastic polarization matrices to capture the far field effects of defects in the material. Our work also focuses on the far field.

2. TWO-DIMENSIONAL ELASTICITY

The plane elasticity stress-strain relations for a linear elastic and isotropic material are given by

$$\varepsilon_{ij} = \frac{1}{2G'}\left[\sigma_{ij} - \kappa \frac{3}{4}\sigma_{kk}\delta_{ij}\right] \quad i, j, k = 1, 2$$

where $\varepsilon_{ij}$ and $\sigma_{ij}$ are the strain and stress tensors, respectively, $G'$ is the shear modulus and $\kappa = 3 - 4v'$ for plane strain and $\kappa = (3 - v')/(1 + v')$ for plane stress where $v'$ is the Poisson's ratio. In this paper we use a primed notation for 3D elastic constants and unprimed for 2D.
Alternately, we can write eqn. (1) as
\[ \varepsilon_{ij} = \frac{1}{E} \left[ (1 + \nu) \sigma_{ij} - \nu \sigma_{kk} \delta_{ij} \right] \quad i, j, k = 1, 2 \]  
(2)

where E and \( \nu \) are a 2D (planar) Young's modulus and a 2D Poisson's ratio, respectively. Note that in 2D, the upper bound on \( \nu \) is 1, as opposed to 1/2 for \( \nu \) in 3D.

In addition, we introduce a 2D (planar) bulk modulus \( K \) and shear modulus \( G \) which are defined in terms of 2D constants as (Thorpe and Sen, 1985)
\[ K = \frac{E}{2(1 - \nu)} \]  
(3)
\[ G = \frac{E}{2(1 + \nu)} \]  
(4)

The other useful relations are
\[ \frac{4}{E} = \frac{1}{K} + \frac{1}{G} \]  
(5)
\[ \nu = \frac{K - G}{K + G} \]  
(6)

A summary of relations between 2D and 3D elastic constants is given in Table 1. In the remaining part of this paper we use the 2D elastic constants.

3. COMPLEX VARIABLE METHOD OF ELASTICITY

First we consider a plane elasticity problem of a single polygonal hole embedded in an isotropic, homogeneous and linear elastic sheet which is subjected to uniform remote tractions. In the analysis we use the complex variable method of elasticity and conformal transformation (Muskheilishvili, 1953; Savin, 1961). This method has been successfully used to solve many problems of practical importance, including those with polygonal geometries, and its great advantage is the flexibility to treat arbitrary shapes. Here we summarize this approach for completeness.

Recall from complex variable theory that if we have two complex domains \( S \) and \( \Sigma \) in the \( z \) and \( \zeta \) planes respectively, the conformal transformation is given by
\[ z = w(\zeta) \]  
(7)

where \( w \) is a holomorphic function and \( \zeta = \rho e^{i\psi} \), where \( \rho \) and \( \psi \) are polar coordinates. If both the domains \( S \) and \( \Sigma \) are infinite, this transformation is of the form
\[ w(\zeta) = B \left[ \zeta + \sum_{n=1}^{\infty} a_n \zeta^{-n} \right] \]  
(8)

If we want to transform the exterior of a unit circle in the \( \zeta \) plane into the exterior of the regular polygon in the \( z \) plane (such that the center of the circle corresponds to the center of the polygon) we use the following transformation
\[ w(\zeta) = B \left[ \zeta + \frac{2}{n(n-1)\zeta^{n-1}} + \frac{n-2}{n^2(2n-1)\zeta^{2n-1}} \right. \]
\[ \left. + \frac{(n-2)(2n-2)}{3n^3(3n-1)\zeta^{3n-1}} \right] \]
\[ + \frac{(n-2)(2n-2)(3n-2)}{12n^4(4n-1)\zeta^{4n-1}} + \ldots \]  
(9)

where \( n = 3, 4, 5, \ldots \) (\( n \) is the number of sides in a polygon), and
\[ B = \text{Re}^{\delta}, \]  
where \( R \) is a real constant and \( \delta \) is the angle rotated from the original position of the polygon. The full transformation for a polygon is of the Schwarz-Christoffel type and eqn. (9) is the expanded form of this transformation. If only first few terms of the series in (9) are used the polygon has rounded off corners as seen in Fig. 1. The parametric equations for the hole can be obtained by setting \( \rho = 1 \) in \( \zeta \) and separating real and imaginary parts. For more details see Muskheilishvili (1953) or Thorpe (1992).

The complex variable method consists of the determination of two analytic functions \( \phi(z) \) and \( \psi(z) \). If we substitute \( z = w(\zeta) \) in \( \phi(z) \) and \( \psi(z) \) we have
\[ \phi(\zeta) = \phi[w(\zeta)] \quad \psi(\zeta) = \psi[w(\zeta)] \]  
(10)

Table 1: Relations between 2D and 3D elastic constants

<table>
<thead>
<tr>
<th>2D elastic constants</th>
<th>Plane stress</th>
<th>Plane strain</th>
<th>General relations</th>
<th>2D relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu )</td>
<td>( \nu' )</td>
<td>( \frac{\nu'}{1-\nu'} )</td>
<td>( \frac{K-G}{K+G} )</td>
<td></td>
</tr>
<tr>
<td>( E' )</td>
<td>( \frac{E'}{1-\nu'^2} )</td>
<td>( 8G' )</td>
<td>( \frac{4}{K+1} )</td>
<td></td>
</tr>
<tr>
<td>( K' )</td>
<td>( \frac{E'}{2(1-\nu')} )</td>
<td>( \frac{E'}{2(1+\nu') (1-2\nu')} )</td>
<td>( \frac{2G'}{\kappa-1} )</td>
<td>( \frac{E}{2(1-\nu)} )</td>
</tr>
<tr>
<td>( G' )</td>
<td>( \frac{E'}{2(1+\nu')} )</td>
<td>( \frac{E'}{2(1+\nu')} )</td>
<td>( G' )</td>
<td>( \frac{E}{2(1+\nu)} )</td>
</tr>
</tbody>
</table>
Here for simplicity of notation we denote \( \phi(z) \) and \( \phi(\zeta) \) by the same symbol even though they represent different functions of the argument. Then, the stresses and displacements in the \( \zeta \) plane can be expressed as

\[
\sigma_{\rho\rho} + \sigma_{\nu\nu} = 4\Re\Phi(\zeta) \tag{11}
\]

\[
\sigma_{\nu\nu} - \sigma_{\rho\rho} + 2i\sigma_{\rho\nu} = \frac{2t^2}{\rho w'(\zeta)} \left[ w(\zeta) \Phi'(\zeta) + w'(\zeta) \Psi(\zeta) \right] \tag{12}
\]

\[
2G|w'(\zeta)| (u_\rho + iu_\nu) = \frac{\zeta}{\rho} w'(\zeta) \left[ \kappa \Phi(\zeta) - \frac{w(\zeta)}{w'(\zeta)} \Phi'(\zeta) - \Psi(\zeta) \right] \tag{13}
\]

where bars denote the conjugate, and \( \Phi(\zeta) \) and \( \Psi(\zeta) \) are defined as

\[
\Phi(\zeta) = \frac{\phi'(\zeta)}{w'(\zeta)} \quad \Psi(\zeta) = \frac{\psi'(\zeta)}{w'(\zeta)} \tag{14}
\]

The boundary conditions on the surface \( \Gamma \) of the hole, given by \( \sigma = e^{i\nu} \) (\( \nu = 1 \)), are in the form

\[
\phi(\zeta) + \frac{w(\sigma)}{w'(\sigma)} \Phi'(\sigma) + \Psi(\sigma) = f(\sigma) \tag{15}
\]

where

\[
f(\sigma) = i\int (\tau_x + i\tau_y) \, dS \tag{16}
\]

and \( \tau_x \) and \( \tau_y \) are tractions on this boundary. In the problems considered here, the surface of the hole is traction free and therefore \( f(\sigma) = 0 \).

For convenience we can write \( \phi(\zeta) \) and \( \psi(\zeta) \) in the form

\[
\phi(\zeta) = A_0 R\zeta + \phi_1(\zeta) \tag{17}
\]

\[
\psi(\zeta) = B_0 R\zeta + \psi_1(\zeta) \tag{18}
\]

where

\[
A_0 = \frac{\sigma_{xx}^0 + \sigma_{yy}^0}{4} \quad B_0 = \frac{\sigma_{xy}^0 - \sigma_{xx}^0 + 2i\sigma_{xy}^0}{2} \tag{19}
\]

and

\[
\phi_1(\zeta) = \sum_{n=1}^{\infty} c_n \zeta^{-n} \quad \psi_1(\zeta) = \sum_{n=1}^{\infty} d_n \zeta^{-n} \tag{20}
\]

In our calculations we consider a uniaxial tension case with \( \sigma_{xx}^0 = \sigma \). Substituting (17)-(18) into the boundary condition (15) we have

\[
\phi_1(\sigma) + \frac{w(\sigma)}{w'(\sigma)} \Phi_1'(\sigma) + \Psi_1(\sigma) = f_1(\sigma) \tag{21}
\]

where

\[
f_1(\sigma) = -A_0 R\sigma - A_0 R \frac{w(\sigma)}{w'(\sigma)} - \frac{B_0 R}{\sigma} \tag{22}
\]

If we multiply each term in eqn. (21) and its conjugate equation by \( 1/[2\pi i (\alpha - \zeta)] \) and integrate over the contour \( \Gamma \), we have two equations for two unknown stress functions \( \phi_1(\zeta) \) and \( \psi_1(\zeta) \)

\[
\phi_1(\zeta) = -\frac{1}{2\pi i} \int \frac{f_1(\sigma)}{\sigma - \zeta} \, d\sigma + \frac{1}{2\pi i} \int \frac{w(\sigma)}{\sigma - \zeta} \Phi_1'(\sigma) \, d\sigma \tag{23}
\]

\[
\psi_1(\zeta) = -\frac{1}{2\pi i} \int \frac{f_1(\sigma)}{\sigma - \zeta} \, d\sigma + \frac{1}{2\pi i} \int \frac{w(\sigma)}{\sigma - \zeta} \Phi_1'(\sigma) \, d\sigma \tag{24}
\]

FIG. 1 Showing the shape corresponding to the conformal mapping given in eqn. (9) with \( n=3 \) (a) and \( n=4 \) (b). The dashed line corresponds to truncating the series after 2 terms and the solid line is the first 3 terms. In the limit of an infinite number of terms a perfect triangle is obtained. For the triangle the area of the dashed figure is 1.0037 that of the perfect triangle attained in the limit as given in eqn. (49); the area of the solid figure is 1.00055 that of the perfect triangle.
Note that the second integral in eqn. (23) contributes when the $\xi^{-s}$ term in the mapping (9) is such that $s \geq 3$, and is zero for an elliptical hole (Sokolnikoff, 1986) and a triangular hole with one term in (9), for example. After the stress functions are found the stresses and displacements can be evaluated using eqns. (11)-(13). We have carried out these calculations with the help of the symbolic manipulation program MAPLE V (1992). If we express these stresses in terms of powers of $1/r$ they are in the form of infinite series.

When a small circular hole (of a unit radius) is embedded in a large plate and subjected to a remote uniaxial tension $\sigma^0_{zz} = T$, the solution is given by the Airy stress function $U$ as follows (Timoshenko and Goodier, 1970)

$$U = \frac{T}{4} \left[ r^2 - r^2 \cos 2\theta + 2 c \log r + 2 d (1 - \nu) \sin 2\theta + \frac{c \cos 2\theta}{r^2} \right]$$  \hspace{1cm} (25)

The stresses $\sigma_{rr}$ and $\sigma_{\theta\theta}$ can be derived from the Airy stress function and are given by

$$\sigma_{rr} = \frac{T}{2} \left[ 1 + \cos 2\theta - \frac{c + 4d \cos 2\theta}{r^2} + \frac{3e \cos 2\theta}{r^4} \right]$$  \hspace{1cm} (26)

$$\sigma_{\theta\theta} = -\frac{T}{2} \left[ \sin 2\theta + \frac{2d \sin 2\theta}{r^2} + \frac{3e \sin 2\theta}{r^4} \right]$$  \hspace{1cm} (27)

$$\sigma_{\theta\theta} = \frac{T}{2} \left[ 1 - \cos 2\theta + \frac{c}{r^2} - \frac{3e \cos 2\theta}{r^4} \right]$$  \hspace{1cm} (28)

where $c = d = e = 1$. When the hole has an arbitrary shape the stresses and displacements involve an infinite number of terms, but the leading terms in the far field are of a similar form

$$\sigma_{rr} = \frac{T}{2} \left[ 1 + \cos 2\theta - \frac{c + 4d \cos 2\theta}{r^2} + \ldots \right]$$  \hspace{1cm} (29)

$$\sigma_{\theta\theta} = -\frac{T}{2} \left[ \sin 2\theta + \frac{2d \sin 2\theta}{r^2} + \ldots \right]$$  \hspace{1cm} (30)

$$\sigma_{\theta\theta} = \frac{T}{2} \left[ 1 - \cos 2\theta + \frac{c}{r^2} - \frac{3e \cos 2\theta}{r^4} + \ldots \right]$$  \hspace{1cm} (31)

$$u_r = \frac{TR}{2E} \left[ (1 - \nu) r + (1 + \nu) r \cos 2\theta + \frac{c (1 + \nu)}{r} + \frac{4d \cos 2\theta}{r} + \ldots \right]$$  \hspace{1cm} (32)

These forms for the far field completely determine the effective elastic moduli of the media. The essential idea is that the hole is replaced by an equivalent circular hole for purposes of obtaining the far field and hence the effective moduli. A similar approach has proved to be straightforward and useful in dielectric problems (Thorpe, 1992). We emphasize that this far field approach is exact and involves no approximations. For objects that lack a symmetry axis for the uniaxial tension a term involving $d' \sin 2\theta$ will also occur in (25). This term also contributes to the far field in stresses and displacements. It does not however contribute to the effective moduli. In the work in this paper, we always align the external stress $T$ along a symmetry axis of the hole, so that $d' = 0$. We will see in the next section, that only the terms with $c$ and $d$ contribute to the elastic moduli. This is expected as those are the leading terms and arise because of the elastic polarizability of the inclusion (Movchan, 1992; Thorpe, 1992). The combination $\sigma_{rr} + \sigma_{\theta\theta}$ in (11) couples to the coefficient $d$, whereas the combination $\sigma_{rr} - \sigma_{\theta\theta}$ from the real part of (12) couples to both $c$ and $d$. The other stress component $\sigma_{\theta\theta}$ is obtained from the imaginary part of (12) and includes $d$. The displacement $u_r$ in (13) couples to the coefficients $c$ and $d$ while $u_\theta$ includes $d$ only. Thus the most straightforward procedure is to calculate either $u_r$ or $\sigma_{\theta\theta} - \sigma_{rr}$ from which both $c$ and $d$ can be obtained as they have different angular dependences and so can be easily identified.

4. DILUTE RESULT

When the concentration of inclusions is very small and there is no interaction between the inclusions, the effective elastic moduli of a material can be calculated exactly. This dilute result can be obtained by using a single inclusion solution and the equivalence of elastic strain energies (Christensen, 1979).

When a remote stress field $\sigma_{ij}^0$ is applied to the domain $D$ containing a single inclusion $Q_{ij}$, the elastic strain energy is expressed by Eshelby’s formula as

$$W = W^0 + \frac{1}{2} \int_{\partial D} (\sigma_{ij}^0 n_i n_j - \sigma_{ij}^0 n_i u_j^0) \, dS$$  \hspace{1cm} (34)

where $W^0 = \frac{1}{2} \int_D \sigma_{ij}^0 \varepsilon_{ij}^0 \, dV$ and $|\partial D|$ is the surface of the inclusion. The superscript zero denotes quantities due to the applied load in the absence of an inclusion. The quantities $\sigma_{ij}^0$ and $u_j^0$ denote the total stresses and displacements, respectively, which include $\sigma_{ij}^0$, $u_j^0$ and the disturbance due to the presence of the inclusions. The unit vector $n_i$ is normal to the inclusion-matrix interface. When the inclusion is a hole, eqn. (34) reduces to

$$W = W^0 + \frac{1}{2} \int_{\partial D} \sigma_{ij}^0 n_i u_j \, dS$$  \hspace{1cm} (35)
The elastic strain energy stored in the equivalent homogeneous medium is

\[ \frac{W}{D} = \frac{1}{2} \epsilon_{ij}^0 c_{ijkl} \epsilon_{kl}^0 \]  \hspace{2cm} (36)

where \( c_{ijkl} \) is the effective compliance. Then, the effective elastic constants of the material with holes can be found by equating the elastic strain energies in eqns. (35) and (36).

In our analysis we use the formula (34) but with the integral taken over some very large surface \( |\Omega_0| \), circular for convenience, of radius \( r_0 \). This way only the stress terms with \( 1/r^2 \) and displacements with \( 1/r \) will remain important and will contribute towards the elastic moduli. This is true for the inclusions of circular and polygonal shapes and is clearly seen when the local fields (29)-(30) and (32)-(33) are substituted in the Eshelby's formula (34) as follows

\[ W = W^0 + \frac{2\pi}{2} \int_0^r \left[ \sigma_{rr}^0 u_r + \sigma_{\theta\theta}^0 u_\theta - (\sigma_{rr}^0 u_r + \sigma_{\theta\theta}^0 u_\theta) \right] r \, dr \, d\theta \]  \hspace{2cm} (37)

When we evaluate the effective 2D Young's modulus \( E^c \), for example, for a circular case we find

\[ \frac{1}{E^c} = \frac{1}{E} (1 + 3f) \]  \hspace{2cm} (38)

where \( f \) is the volume (area) fraction of holes given by \( f = A_{\text{circle}} / A_{\text{total}} \) with \( A_{\text{circle}} = \pi r_0^2 \) for the unit circle, and \( E \) is the 2D elastic modulus of the matrix material.

The expression for the effective 2D Young's modulus of a material with polygonal holes can be obtained in a similar way and is of the form

\[ \frac{1}{E^c} = \frac{1}{E} \left[ 1 + (2d + c) \frac{\pi}{A_{\text{poly}}} f \right] \]  \hspace{2cm} (39)

where \( f = A_{\text{poly}} / A_{\text{total}} \) is the area fraction of polygons. The expression (39) reduces to the expression for the circle with \( c = d = 1 \), and replacing \( A_{\text{poly}} \) with \( A_{\text{circle}} = \pi r_0^2 \).

In Jasiuk et al (1992) we expressed the effective 2D elastic modulus \( E^c \) and the effective 2D Poisson's ratio \( \nu^c \) of a material with dilute concentration of randomly distributed holes as

\[ E^c = E (1 - \alpha f) \]  \hspace{2cm} (40)

\[ \nu^c = \nu - \alpha (\nu_0 - \nu) f \]  \hspace{2cm} (41)

where the parameters \( \alpha \) and \( \nu_0 \), given in Table 2, depend only on the shape of the polygon and not on the elastic constants of the matrix. Here, we express these two variables in terms of the far field constants \( c \) and \( d \) as follows

\[ \alpha = (2d + c) \frac{A_{\text{circle}}}{A_{\text{poly}}} \]  \hspace{2cm} (42)

\[ \nu_0 = \frac{2d - c}{2d + c} \]  \hspace{2cm} (43)

The effective 2D bulk and shear moduli are given by

\[ \frac{K^c}{K} = 1 - \alpha \frac{1 - \nu_0 f}{1 - \nu} \]  \hspace{2cm} (44)

\[ \frac{G^c}{G} = 1 - \alpha \frac{1 + \nu_0 f}{1 + \nu} \]  \hspace{2cm} (45)

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Table 1: Analytical and numerical values for parameters \( \alpha \) and \( \nu_0 \)

<table>
<thead>
<tr>
<th></th>
<th>Analytical (one term)</th>
<th>Analytical (two terms)</th>
<th>Analytical (three terms)</th>
<th>Numerical Simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>( \alpha = 4.1429 )</td>
<td>( \nu_0 = 0.2414 )</td>
<td>( \alpha = 4.1897 )</td>
<td>( \nu_0 = 0.2332 )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 4.2019 )</td>
<td>( \nu_0 = 0.2312 )</td>
<td>( \alpha = 4.2019 )</td>
<td>( \nu_0 = 0.227 \pm 0.003 )</td>
</tr>
<tr>
<td>Square</td>
<td>( \alpha = 3.4259 )</td>
<td>( \nu_0 = 0.3101 )</td>
<td>( \alpha = 3.4580 )</td>
<td>( \nu_0 = 0.3074 )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 3.4672 )</td>
<td>( \nu_0 = 0.3067 )</td>
<td>( \alpha = 3.4672 )</td>
<td>( \nu_0 = 0.302 \pm 0.006 )</td>
</tr>
<tr>
<td>Pentagon</td>
<td>( \alpha = 3.2092 )</td>
<td>( \nu_0 = 0.3249 )</td>
<td>( \alpha = 3.2285 )</td>
<td>( \nu_0 = 0.3235 )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 3.2342 )</td>
<td>( \nu_0 = 0.3238 )</td>
<td>( \alpha = 3.2342 )</td>
<td>( \nu_0 = 0.325 \pm 0.003 )</td>
</tr>
<tr>
<td>Hexagon</td>
<td>( \alpha = 3.1186 )</td>
<td>( \nu_0 = 0.3295 )</td>
<td>( \alpha = 3.1306 )</td>
<td>( \nu_0 = 0.3289 )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 3.1342 )</td>
<td>( \nu_0 = 0.3291 )</td>
<td>( \alpha = 3.1342 )</td>
<td>( \nu_0 = 0.328 \pm 0.003 )</td>
</tr>
</tbody>
</table>
We call \( v_0 \) the invariant Poisson's ratio because if the Poisson's ratio of the matrix \( \nu \) equals \( v_0 \), the linear term in \( f \) vanishes in (41) and the effective Poisson's ratio \( \nu^c \) does not change (Jasiuk et al., 1992). From Hashin's second order bounds (Hashin, 1965) on \( E^c \) it can be shown that \( 3 \leq \alpha < \infty \), where the lower and upper bounds on \( \alpha \) correspond to circular and needle-like holes, respectively. Comparing the values of \( \alpha \) in Table 1 we see that the materials with polygonal holes are more compliant than those having circular holes and the triangular (equilateral) shape yields the lowest elastic modulus \( E^c \) as compared with other regular polygons.

For a material containing a few randomly oriented ellipses the coefficients \( c \) and \( d \) can be found explicitly. Averaging over two orientations at right angles (which is equivalent to an isotropic average) we have

\[
c = \frac{1}{2} (a^2 + b^2) \quad d = \frac{1}{4} (a + b)^2
\]  

(46)

where \( a \), \( b \) are the semi-major and semi-minor axes of the ellipse. From this far field result we find that

\[
\alpha = \frac{1}{v_0} = 1 + \frac{a}{b} + \frac{b}{a}
\]  

(47)

as given in Thorpe and Jasiuk (1992).

The area of the polygon can be obtained by separating (9) into real and imaginary parts and integrating

\[
A = \int dxdy = \frac{1}{2} \int (x dy - y dx) \\
= \frac{1}{2} \int [xy' (\theta) - xy' (\theta)] d \theta \\
= \pi \left[ \frac{1}{n} \frac{4}{n^2 (n-1)} - \frac{(n-2)^2}{n^4 (2n-1)} - \frac{(n-2)^2 (2n-2)^2}{9n^6 (3n-1)} - \ldots \right]
\]

(48)

Summing the infinite series we find (Thorpe, 1992) that \( A \) is

\[
A = \frac{(2\pi)^2 n}{\tan \frac{\pi}{n}} \frac{\Gamma^2 (\frac{2}{n})}{\Gamma^4 (\frac{1}{n})}
\]  

(49)

As the higher order terms are added in eqn. (48), higher order harmonics are introduced which have the effect of straightening the sides and sharpening the corners as shown in Fig. 1. It is interesting to note that retaining the first two terms in the series (48) gives an area to within 1% of the area of a perfect polygon.

We can also use eqn. (35) together with (36) to evaluate the effective elastic moduli of materials with holes. This is a near field approach which has been used by Zimmerman (1986) and Kachanov et al. (1994). In principle, these two methods yield the same results for the elastic moduli but we have found the far field approach to be simpler for isotropic holes.

5. CLM theorem

A new result in plane elasticity, which we refer to as the CLM theorem, has recently been proved by Cherkaev, Lurie and Milton (Cherkaev et al., 1992). This result applies to 2D linear elastic materials with general anisotropy and arbitrary geometry. It reduces a parameter space of the local stress fields and effective elastic moduli. The results for stresses are related to Dandurs constants (1967, 1969) which reduce the number of independent material constants from three to two for an isotropic and linear elastic two phase material subjected to traction boundary conditions. Also Michell's (1899) solution states that the stress field in a material containing holes and subjected to traction boundary conditions with zero resultant forces over each hole is independent of the elastic constants of the material. The CLM theorem can be stated as follows. Suppose that a 2D composite material with general anisotropy has spatially varying compliances \( S_{ijkl} (e) \). Then, if we consider a transformed material with compliance \( S'_{ijkl} (e) \) given by

\[
S'_{ijkl} (e) = S_{ijkl} (e) + D_{ijkl}
\]

(50)

the stress field remains unchanged in the original and the transformed composite systems and the effective elastic compliances of these two systems are related by

\[
S^{C}_{ijkl} = S_{ijkl}^C + D_{ijkl}
\]

(51)

where \( S_{ijkl}^{C} \) is a general 2D effective anisotropic elastic compliance and \( D_{ijkl} \) is defined as

\[
D_{ijkl} = \frac{C}{2} \delta_{ij} \delta_{kl} - \frac{1}{2} \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \delta_{ij} \delta_{kl} \right)
\]

(52)

If we use contracted form of the fourth rank tensor \( S_{ijkl} \) such that \( 11 \rightarrow 1, 22 \rightarrow 2, \) and \( 12 \rightarrow 3 \) then we can write the constitutive equations in the form

\[
\varepsilon_i = S_{ij} \sigma_j \quad i, j = 1, 2
\]

(53)

where \( \varepsilon_i = \{ \varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12} \}^T \), \( \sigma_i = \{ \sigma_{11}, \sigma_{22}, \sigma_{12} \}^T \).
We can express the results (50) and (51) in the matrix form, and (51) can be written

\[
\begin{bmatrix}
S_{11}^c & S_{12}^c & S_{13}^c \\
S_{12}^c & S_{22}^c & S_{23}^c \\
S_{13}^c & S_{23}^c & S_{33}^c
\end{bmatrix}
= \begin{bmatrix}
S_{11} & S_{12} & S_{13} \\
S_{12} & S_{22} & S_{23} \\
S_{13} & S_{23} & S_{33}
\end{bmatrix}
+ C
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  

(54)

or in components,

\[
S_{11}^c = S_{11} + 2S_{12} + C \\
2S_{12}^c = 2S_{12} + C \\
S_{13}^c = S_{13} + C \\
S_{22}^c = S_{22} - S_{33} - C
\]  

(55)

Equations (54)-(55) also hold for eqn. (50) when we omit the superscript c. As the discussion that follows next applies to both the local properties and effective properties we omit the superscript c.

For the most general anisotropic situation in 2D we have six independent constants, five of which are invariant under the CLM theorem as seen from (55) and (56)

\[
2S_{12}^c + S_{33}^c = 2S_{12} + S_{33}
\]  

(56)

When a material is orthotropic, then

\[
S_{13} = S_{23} = 0
\]  

(57)

and only four independent elastic constants are present with three invariant quantities.

Orthotropic materials have rectangular symmetry and it is useful to consider their behavior when subject to loading in a general direction given by the direction cosine \((n_1, n_2)\), where \(n_1^2 + n_2^2 = 1\). If a tensile stress \(T\) is applied along \((n_1, n_2)\) then the stress tensor is given by

\[
\sigma_{ij} = Tn_in_j
\]  

(58)

and the strain components are therefore

\[
\varepsilon_{11} = T(S_{11}n_1^2 + S_{12}n_2^2)
\]

\[
\varepsilon_{22} = T(S_{12}n_1^2 + S_{22}n_2^2)
\]  

(59)

\[
\varepsilon_{12} = TS_{33}n_1n_2/2
\]

The longitudinal strain is given by

\[
\varepsilon_L = \varepsilon_{ij}n_in_j
\]  

(60)

so that combining (59) and (60) we have

\[
\varepsilon_L = T(S_{11}n_1^4 + 2S_{12}n_1^2n_2^2 + S_{22}n_2^4 + S_{33}n_1^2n_2^2)
\]  

(61)

and we can define a Young's modulus \(E\) along \((n_1, n_2)\) by

\[
\varepsilon_L = T/E
\]

to give

\[
\frac{1}{E} = S_{11}n_1^4 + 2S_{12}n_1^2n_2^2 + (2S_{12} + S_{33})n_2^4
\]  

(62)

Thus there are now three independent moduli given by

\[
\frac{1}{E_1} = S_{11} \\
\frac{1}{E_2} = S_{22} \\
\frac{1}{E_3} = S_{12} + \frac{S_{33}}{2}
\]  

(63)

and in the isotropic limit, when all directions in the material are equivalent, we have \(E_1 = E_2 = E_3\). Note that \(E_3\) is not really a Young's modulus, as it cannot be found in a single measurement for any \((n_1, n_2)\) but we use this notation for convenience.

An anisotropic Poisson's ratio can also be defined by using the transverse strain \(\varepsilon_T\) along \((n_2, -n_1)\) which is the direction perpendicular to \((n_1, n_2)\). Hence

\[
\varepsilon_T = \varepsilon_{11}n_2^2 + \varepsilon_{22}n_1^2 - 2\varepsilon_{12}n_1n_2
\]  

(64)

and when we combine (59) and (64) we have

\[
\varepsilon_T = T\left[(S_{11} + S_{22} - \frac{S_{33}}{2})n_1^2n_2^2 + S_{12}(n_1^4 + n_2^4)\right]
\]  

(65)

Therefore from (61) and (65), we can define a Poisson's ratio \(\nu = -\varepsilon_{21}/\varepsilon_L\) associated with the direction \((n_1, n_2)\).

It is sometimes useful to calculate isotropically averaged quantities, and this can be done most simply from equations (61) and (64) by using

\[
\langle n_1^4 \rangle = \langle n_2^4 \rangle = \frac{3}{8} \\
\langle n_1n_2^2 \rangle = \frac{1}{8}
\]  

(66)

where the brackets \(\langle \rangle\) denote an angular average. We obtain the isotropically averaged compliance \(\frac{1}{E}\) from (62) as
\begin{equation}
\frac{1}{E} = \frac{3}{8} (S_{11} + S_{22}) + \frac{1}{4} \left( S_{12} + \frac{S_{33}}{2} \right) = \frac{3}{8} \left( \frac{1}{E_1} + \frac{1}{E_2} \right) + \frac{1}{4} \frac{1}{E_3} \tag{67}
\end{equation}

This is equivalent to averaging the longitudinal strain in eqn. (61). In a similar way we can average the transverse strain in eqn. (64). By dividing the averaged transverse strain by the averaged longitudinal strain, we can define averaged Poisson's ratio \( \langle \nu \rangle \) by

\begin{equation}
\langle \nu \rangle = -\left[ \frac{S_{11} + S_{22} + 6S_{12} - S_{33}}{3S_{11} + 3S_{22} + 2S_{12} + S_{33}} \right] \tag{68}
\end{equation}

Note that the numerator and denominator have been averaged separately in order to give a tractable expression. The averaging procedure is therefore best thought of in terms of averaging the longitudinal and transverse strains.

From our previous discussion of the CLM theorem, we see that \( E_1, E_2 \), and \( E_3 \) in (63) are all invariants. If holes are cut in the material then the general form for all three would be

\begin{equation}
E_i^c = E h_i(f) \quad i = 1, 2, 3 \tag{69}
\end{equation}

where \( E_i^c \) are the effective moduli and \( E \) is the Young's modulus of the isotropic host material. The functions \( h_i(f) \) depend only on the geometry associated with the anisotropic arrangement of holes and not upon the Poisson's ratio \( \nu \) of the host material.

For the fourth quantity required for rectangular symmetry we choose \( \langle \nu \rangle \) from (68). Under the CLM transformation (51), it has the property that

\begin{equation}
\langle \nu^c \rangle = \langle \nu \rangle - \frac{C}{2} \langle \frac{1}{E^c} \rangle \tag{70}
\end{equation}

so that we may write in general

\begin{equation}
\langle \nu^c \rangle = h_4(f) + \langle \nu \rangle - \frac{1}{3} E \langle \frac{1}{E^c} \rangle \tag{71}
\end{equation}

and the fourth function \( h_4(f) \) is also independent of the Poisson's ratio of the host material. Any convenient reference value of Poisson's ratio can be taken in (71). We have chosen \( 1/3 \) as this is convenient for our numerical simulations. Any other value would suffice; the only difference being that \( h_4(f) \) would be altered to a new function \( h_4'(f) \), also independent of \( \nu \). The function \( h_4'(f) \) would be a linear combination of all the \( h_i(f) \).

These results are needed to obtain the isotropically averaged results for the square hole given in Table 2 and compared with the numerical simulation results in Jasiuk et al. (1992). Using the conformal mapping described earlier, we obtain two sets of results; for the tension \( T \) along a body diagonal of a square (which we denote by [11] direction) and for the tension \( T \) along a line through the center of two edges (which we denote by [10] direction). This leads to four constants \( c_{[10]}^{[10]}, d_{[10]}^{[11]}, c_{[11]}^{[10]} \) and \( d_{[11]}^{[11]} \) (and also the related constants \( \alpha_{[10]}^{[10]}, \nu_{[10]}^{[10]} \), \( \alpha_{[11]}^{[11]}, \nu_{[11]}^{[11]} \)). From eqn. (62), we see that

\begin{equation}
\frac{1}{E_{[10]}} = S_{11} = \frac{1}{E_1} \tag{72}
\end{equation}

and

\begin{equation}
\frac{1}{E_{[11]}} = \frac{S_{11}}{2} + \frac{2S_{12} + S_{33}}{4} = \frac{1}{2} \left[ \frac{1}{E_1} + \frac{1}{E_3} \right] \tag{73}
\end{equation}

and of course \( S_{11} = S_{22} \) (and also \( E_1 = E_2 \)) for square symmetry. Thus from (67), we have

\begin{equation}
\langle \frac{1}{E} \rangle = \frac{1}{2} \left( \frac{1}{E_{[10]}} + \frac{1}{E_{[11]}} \right) \tag{74}
\end{equation}

and hence the isotropically averaged \( \alpha \) is given by

\begin{equation}
\alpha = \frac{1}{2} \left( \alpha_{[10]}^{[10]} + \alpha_{[11]}^{[11]} \right) \tag{75}
\end{equation}

Extending these arguments to the transverse strain and hence the isotropically averaged Poisson's ratio \( \langle \nu \rangle \) given in (68), we find that the correct procedure is to average the \( c \) and \( d \) coefficients

\begin{equation}
c = \frac{1}{2} \left( c_{[10]}^{[10]} + c_{[11]}^{[11]} \right) \quad d = \frac{1}{2} \left( d_{[10]}^{[10]} + d_{[11]}^{[11]} \right) \tag{76}
\end{equation}

and hence from (43) we find that in addition to (75) we also have

\begin{equation}
\langle \nu^c \rangle = \frac{\alpha_{[10]}^{[10]} \nu_{[10]}^{[10]} + \alpha_{[11]}^{[11]} \nu_{[11]}^{[11]}}{\alpha_{[10]}^{[10]} + \alpha_{[11]}^{[11]}} \tag{77}
\end{equation}

These results were used for the square in Table 2.

6. ANISOTROPIC MODULI - ELLIPTICAL HOLES

As an illustration of the power of the CLM theorem, we consider the case of a sheet containing elliptical inclusions. These
inclusions are all of the same shape, with major semi-axis $a$ and minor semi-axis $b$. In order to create an anisotropic system, we align these ellipses with the centers located randomly. A number of exact results can be proved for this system. From Eshelby's three dimensional solution (Eshelby, 1957) for a general ellipsoid, it is possible to let one of the axis become infinity and hence obtain the solution for a two dimensional ellipse. We can also use the complex variable method discussed earlier. From the solution for the single elliptical hole, the elastic compliances can be obtained in the standard way. This was done previously by Thorpe and Sen (1985) for randomly oriented ellipses. Here we omit the orientational averaging to obtain the elastic compliances.

FIG. 2. A plot of the various elastic compliances for a sheet containing randomly oriented aligned ellipses versus the area fraction of holes $f$. The symbols are simulation results for an aspect ratio $a/b=2$, where $2a = 16$ and $2b = 8$ on a $210 \times 210$ triangular lattice. Simulation results are averaged over five different configurations. The dashed lines are the theoretical predictions in dilute limit from eqns. (78)-(80).

FIG. 3. The inset shows the Poisson's ratio for a sheet containing randomly oriented ellipses versus the area fraction of the holes $f$. The symbols are simulation results for an aspect ratio $a/b=2$, where $2a = 16$ and $2b = 8$ on a $210 \times 210$ triangular lattice. Simulation results are averaged over five different configurations. The dashed line is the theoretical prediction in the dilute limit from eqn. (61). The Young's modulus results are from the isotropic average [see eqn. (67)] and the dashed line is the theoretical prediction in the low concentration limit. The solid line is the fitted interpolation formula for the Young's modulus for randomly centered circular holes [Day et al, 1992, eqn (32)].
\[ \frac{S_{11}^c}{S_{11}} = 1 + f \left( 1 + \frac{2b}{a} \right) \quad (78) \]

\[ \frac{S_{22}^c}{S_{22}} = 1 + f \left( 1 + \frac{2a}{b} \right) \quad (79) \]

\[ \frac{2S_{12} + S_{33}^c}{2S_{12} + S_{33}} = 1 + f \left( 1 + \frac{a}{b} + \frac{b}{a} \right) \quad (80) \]

These expressions are exact in the dilute limit of small \( f \), and are independent of the Poisson's ratio of the matrix material as required by the CLM theorem. For this rectangular symmetry there are four independent elastic moduli. These results are consistent with the general results given in the previous section. From our numerical simulations, we choose to display \( S_{11}, S_{22} \) and the isotropically averaged \( \left\langle \frac{1}{2} \right\rangle \) which is a linear combination of (78)-(80) as shown in (67).

For our fourth quantity, we choose the isotropically averaged Poisson's ratio \( \left\langle \nu^c \right\rangle \) given in (68) for which the result in the dilute limit is

\[ \left\langle \nu^c \right\rangle = \nu - \left[ (1 + \frac{a}{b} + \frac{b}{a}) \nu - 1 \right] f \quad (81) \]

which is the same as given in eqn. (16) of Jasiuk et al (1992) for randomly oriented ellipses.

Our simulations are performed using the spring grid scheme as described in Day et al (1992). Because of the CLM theorem we only need to use a single value of \( \nu \) to determine the four geometrical functions \( h_1(f) \) that completely characterize the elastic behavior of the system. In order to compute all four quantities of interest, it was necessary to put strains successively in (1,0), (0,1) and (1,1) directions and determine the Young's modulus and Poisson's ratio in each case. This led to the determination of six quantities which was more than the four needed, but was useful in providing internal consistency checks. Our results are shown in Figs. 2 and 3. The error bars are from the statistical spread between the five random geometries used.

The area fraction at percolation for aligned ellipses is the same as for circles as can easily be seen by uniformly deforming the samples so that the ellipses turn into circles to give a critical area fraction \( f_c = 0.66 \pm 0.03 \) (Day et al, 1992).

In Fig. 2 we see that the three Young's moduli all decrease rapidly as \( f \) is increased. The simulations agree well with the exactly known initial slopes given in (78)-(80). The quantity \( S_{11} \) has the smallest initial slope, because the applied strain is along the major axis of the ellipse. As the ellipse approaches a needle shape, this slope would approach zero for all concentrations. The solid line is the result for randomly centered circular holes from Day et al (1992). It is an interpolation formula that is fit to the numerical results and is included for reference. It is clear that \( \frac{E^c}{E} \) is always just below the result for circles.

The elastic moduli go to zero at the critical point with a large exponent and so it is impossible to get simulation results for \( f \) greater than about 0.5, which is quite far from the critical concentration of 0.66. The error bars on the isotropically averaged Poisson's ratio become quite large as \( f \) is increased. This was also found by Day et al (1992) for circles because two small quantities are being divided as the elastic moduli decrease rapidly as \( f \) increases. It is hard to judge where the critical value of \( \nu^c \) is, but it appears to be heading towards a value between 0 and 0.1. From eqn. (81), the initial slope would be flat for

\[ \nu = (1 + \frac{a}{b} + \frac{b}{a})^{-1} = \frac{2}{7} = 0.286. \quad (82) \]

We stress that the simulations have been done for a host Poisson's ratio of \( \nu = 1/3 \) as this allows the first three moduli to be made equal in the spring grid scheme and hence saves computer time. We have checked that the Young's modulus results in Fig. 2 are indeed independent of \( \nu \), and that the results for the Poisson's ratio in Fig. 3 are consistent with the CLM theorem in the form of eqn. (71). These checks were done by using another value of \( \nu \) for a few selected values of area fraction \( f \).

ACKNOWLEDGMENTS

This research was supported by the Composite Materials and Structures Center at Michigan State University through the Research Excellence Fund from State of Michigan. We would like to thank AR Day, E Garboczi and E Tsuchida for useful discussions.

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