ELASTIC PROPERTIES OF TWO-DIMENSIONAL COMPOSITES CONTAINING POLYGONAL HOLES

I. Jasiuk
Department of Materials Science and Mechanics
Michigan State University
East Lansing, Michigan

J. Chen and M. F. Thorpe
Department of Physics and Astronomy
Center for Fundamental Materials Research
Michigan State University
East Lansing, Michigan

ABSTRACT

Evaluation of the effective elastic moduli of two dimensional materials containing polygonal holes reveals that the area Young’s modulus is independent of the Poisson’s ratio of the host material. In addition, the elastic response of materials containing elliptical holes and planar cracks is presented.
1. INTRODUCTION

We discuss some new exact results in two dimensional (plane) elasticity that have applications in studies of the elastic response of two dimensional composite materials. These results are related to a recently proven theorem (Cherkaev, Lurie and Hilton, 1992) which reduces a parameter space of the local stress fields and the effective elastic properties. This theorem is most powerful for materials containing holes. Recently, we have shown that the effective elastic modulus of a sheet containing circular holes is independent of the Poisson's ratio of the matrix (Day et al., 1992; Thorpe and Jasiuk, 1992). Now we consider a material with polygonal holes and also show that the effective elastic modulus depends on the shape of holes but not on the Poisson's ratio of the matrix. In addition, we discuss the results available in literature for the elastic response of materials containing circular and elliptical holes, and planar cracks, which relate directly to the subject of this paper.

2. TWO-DIMENSIONAL ELASTICITY

The stress-strain equations for a linear elastic and isotropic material in three dimensions (3d) are given by (Timoshenko and Goodier, 1951)

\[ \epsilon_{ij} = \frac{1}{E} [(1 + 2\nu)\sigma_{ij} - \nu \sigma_{kk} \delta_{ij}] \quad i,j,k = 1,2,3 \]

(1)

where \( \epsilon_{ij} \) and \( \sigma_{ij} \) are the strain and stress tensors respectively, and \( E \) is the 3d Young's modulus and \( v \) is the 3d Poisson's ratio. We use the primes for the elastic constants in three-dimensions (3d) so that we may use unprimed quantities in two-dimensions (2d).

In 2d elasticity the constitutive equations have a form similar to (1), but only involve two coordinates

\[ \epsilon_{ij} = \frac{1}{E} [(1 + 2\nu)\sigma_{ij} - \nu \sigma_{kk} \delta_{ij}] \quad i,j,k = 1,2 \]

(2)

The unprimed quantities \( E \) and \( V \) are the area Young's modulus and the area Poisson's ratio, respectively. Note that the area Poisson's ratio is bounded by \(-1 < V < 1\), in contrast to the bounds \(-1 < V < 1/2\) for the 3d Poisson's ratio. The area bulk modulus \( K \) and the shear modulus \( G \) are expressed in terms of \( E \) and \( V \) as (Sen and Thorpe, 1985),

\[ K = \frac{E}{2(1-V)} \]

\[ G = \frac{E}{2(1+V)} \]

so that

\[ \frac{4}{E} = \frac{1}{K} + \frac{1}{G} \]

(3)

In this paper we will use equations (2) and the various 2d elastic moduli (3). The 2d constitutive equations (2) are usually derived from (1) by assuming either plane strain or plane stress. We summarize the mappings from 3d to 2d for both plane strain and plane stress in Table 1. The 2d elastic constants discussed in this paper may represent the effective in-plane elastic constants of a transversely isotropic material or a sheet containing inclusions, for example.
Table 1. The connection between the 2d elastic constants (unprimed) and the 3d elastic constants (primed) for both plane strain and plane stress.

<table>
<thead>
<tr>
<th>Plane Strain</th>
<th>Plane Stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K = K + \frac{G}{3} )</td>
<td>( K = \frac{9K}{3K + 4G} )</td>
</tr>
<tr>
<td>( G = G )</td>
<td>( G = G )</td>
</tr>
<tr>
<td>( E = \frac{E}{1-\nu^2} )</td>
<td>( E = E )</td>
</tr>
<tr>
<td>( V = \frac{V}{1-\nu} )</td>
<td>( V = V )</td>
</tr>
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</table>

3. CLM TRANSFORMATION AND THEOREM

A new result in plane elasticity which we refer to as the CLM theorem has recently been proved for 2d composite materials (Cherkaev, Lurie and Milton, 1992). It is based on an earlier work of Lurie and Cherkaev (1986). The CLM theorem applies to linear elastic materials with general anisotropy and holds for an arbitrary geometry. In this paper we are only concerned with the case where the components are isotropic, when the CLM theorem can be stated as follows.

Suppose that a 2d composite material has spatially varying bulk and shear moduli given by \( K(r) \) and \( G(r) \) respectively, and that the effective moduli of the material are \( K' \) and \( G' \), then if

\[
\frac{1}{K'(r)} = \frac{1}{K(r)} - \frac{1}{C} \quad \text{and} \quad \frac{1}{G'(r)} = \frac{1}{G(r)} + \frac{1}{C}
\]

then,

\[
\frac{1}{K^*} = \frac{1}{K} - \frac{1}{C} \quad \text{and} \quad \frac{1}{G^*} = \frac{1}{G} + \frac{1}{C}
\]

where the superscript \( t \) denotes the transformed system and \( C \) is a constant. Note that the constant \( C \) is restricted in order to ensure that the elastic moduli are positive everywhere in the transformed system. The vector \( r \) lies in the plane of the 2d material. We will refer to (4) as the CLM transformation, leading to the CLM theorem (5). Under the CLM transformation the stress field is the same in both the original and the transformed material, for given external tractions, even though the elastic constants differ. Cherkaev, Lurie and Milton (1992) refer to such materials as equivalent. The CLM theorem contains most other previously known exact results in 2d for composite systems, as special cases (Cherkaev, Lurie and Milton, 1992; Thorpe and Jasik, 1992). In this paper we are interested in the case where the elastic moduli are piecewise constant and the composite contains two phases only (which is a special case of (5) and (6)). The CLM theorem applies to arbitrary geometries, but in this paper we are only concerned with polygonal and elliptical holes. The constraint on the allowed values of \( C \) means that the CLM theorem is most powerful in the limit when the inclusions are holes, and becomes useless in the limit where the sample contains rigid inclusions which require that \( C=0 \). In this paper we focus our attention on the case when the inclusions are holes.

It is useful to rewrite the CLM transformation (4) in terms of the Young's modulus using (3) to give
which indicates that the Young's moduli of the material are invariant under the CLM transformation and to rewrite (5) as

\[ E^* = E \]

which states that the area Young's modulus of the composite is invariant under the CLM theorem.

4. DUNDURS CONSTANTS

The CLM transformation is related to an earlier result which is due to Dundurs (1967, 1970). Dundurs showed that if a composite material consisting of two linearly elastic and isotropic phases is subjected to specified tractions and undergoes plane deformation, then the stress has a reduced dependence on the elastic constants. The stress depends only on two dimensionless parameters, as opposed to three dimensionless combinations of elastic constants in 3d. This can be written as

\[ \sigma_{ij} = \sigma_{ij}^* (r, \sigma_{12}, B_{12}) \]

where the vector \( r \) lies in the plane of the 2d material. The Dundurs constants \( \sigma_{12} \) and \( B_{12} \) are defined as

\[ \sigma_{12} = \frac{1 - \frac{1}{E_1 E_2}}{E_1 E_2} \]

\[ B_{12} = \frac{1 - \frac{1}{E_1 E_2}}{K_1 K_2} \]

where the subscripts 1 and 2 denote the matrix material and inclusions, respectively. The Dundurs result follows from the CLM transformation (4), which reduces the number of parameters by one. The parameters \( \sigma_{12} \) and \( B_{12} \) are appropriate as they are clearly invariant under the CLM transformation. The Dundurs constants are not unique and other representations can be used.

A particularly important special case occurs when one of the components, say 2, becomes holes (of any size or shape). Then \( K_2 = E_2 = 0 \), leading to \( \sigma_{12} = -1 \) and \( B_{12} = -E_1/K_1 = -2(1-v_2) \). For holes, the dependence on Poisson's ratio remains in the Dundurs constants, but drops out in the expressions for stresses as expected from the result of Michell (1899). Michell showed that the stresses in a 2d multiply-connected body, induced by specified tractions, are independent of the elastic constants if the resultant force over each boundary vanishes and there are no body forces. This result led to the development of the photoelasticity method, an experimental technique of using optical birefringence in transparent materials to measure the stress fields. These stress fields depend only on the geometry of the holes and not on the elastic constants of the material.

5. A MATERIAL CONTAINING HOLES

If a 2d composite material is made by removing material to form holes (of any size, shape, area fraction etc.), then the relative area Young's modulus
F'/E of a 2d material containing holes is the same for all materials, independent of Poisson's ratio, for a prescribed geometry (Day et al., 1992). This result is easily proved by the CLM result (Cherkaev, Lurie and Milton, 1992; Thorpe and Jasiuk, 1992). The CLM transformation leaves holes as holes, and therefore any matrix material can be reached by using the CLM transformation. Also, the effective area Young's modulus remains unaffected by the change in the Poisson's ratio as seen from (7).

To illustrate this special case of the CLM theorem, we recently considered a sheet with circular holes in various regular and random arrangements and used a computer simulation (Day et al., 1992). For simplicity these holes were all of the same size. We showed that the results all lie on a single curve, independent of the Poisson's ratio of the matrix material, so that

$$E'_{\text{eff}} / E = E' / E$$

(10)

where $E$ is the area Young's modulus of the matrix. The area Poisson's ratio of the composite can be written as

$$\nu'_{\text{eff}} - \nu = (\nu' - \nu) \frac{E}{E}$$

(11)

where we have used the values of the area Poisson's ratio $\nu$ and Young's modulus $E$ of the matrix at the area fraction of holes $f=0$ to eliminate the (unknown) constant C. This equation provided the first rigorous proof of the conjectured flow (Garboczi and Thorpe, 1985; Schwartz et al., 1985; Garboczi and Thorpe, 1986) of the Poisson's ratio to a fixed point at the percolation threshold (Day et al., 1992; Thorpe and Jasiuk, 1992). The result (11) shows explicitly that $\nu'_{\text{eff}} = \nu = E / E=0$. We have found numerically that for randomly centered circular holes with $\nu=1/3$, the Poisson's ratio remains as 1/3 to within numerical accuracy for all values of $f$ (Day et al., 1992). It should be emphasized that the universal value of the area Poisson's ratio is only with respect to the elastic constants of the material; different geometries (arrangement and shape of inclusions) will lead to different universal values of the Poisson's ratio as percolation is approached.

6. A MATERIAL WITH POLYGONAL HOLES

Next, we consider a 2d material with regular polygonal holes (as shown in Fig. 1 for triangles) and investigate the relation between the shape of polygon and the effective elastic moduli in the dilute limit.

Fig. 1 A sketch of a composite containing randomly positioned triangular inclusions.
When the area fraction of polygons \( f \) is small, the effective Young's modulus \( E^* \) can be written as

\[
E^* = 1 - \alpha f
\]

(12)

where \( E \) is the Young's modulus of the matrix and \( \alpha \) is some positive constant which is a function of the shape of polygon only. From the CLM theorem, we know that for a given shape of polygon holes in a matrix, \( E^*/E \) is a universal curve which is independent of the Poisson's ratio of the matrix. The relation (11) holds, so that to leading order in the area fraction \( f \) we have

\[
v^* = v + \beta f
\]

(13)

where \( \beta \) is some constant depending upon the shape of the polygon only. By using (11), (12), and (13), we have

\[
v^{t*} = v^t + \beta f \{ \beta - \alpha (v^t - v) \}
\]

(14)

Therefore, if we choose

\[
v^t = v_0 = v + \frac{\beta}{\alpha}
\]

(15)

then \( v^* = v \) to the leading order in \( f \) so that the term linear in \( f \) vanishes. We shall refer to this value of the Poisson's ratio \( v_0 \) as the invariant Poisson's ratio, and the general result for the Poisson's ratio for small \( f \) can be written

\[
v^* = v - \alpha (v - v_0) f
\]

(16)

For the rest of the paper we will use the notation of (12) and (16). This means that if the matrix has an invariant Poisson's ratio given by (15), then the effective Poisson's ratio does not change (to the leading order in \( f \)) when a small concentration of polygon holes is cut in the matrix. If \( v > v_0 \), then the Poisson's ratio decreases with \( f \), and vice versa, so that the Poisson's ratio tends to "flow towards" \( v_0 \) as \( f \) increases.

The purpose of this work is to study the dependence of constants \( \alpha \) and \( v_0 \) on the shape of polygon. One of the extreme cases is when the polygon is a circular hole. In this case, \( \alpha = 3 \) and \( v_0 = 1/3 \), which can be calculated exactly using the result for a single circular inclusion. In the other cases, no analytical formulas are available yet. But from the Hashin's (Hashin, 1965) second order bounds on \( E^* \), we can show that \( 3 \leq \alpha \leq \infty \). We note that these bounds are attained for the circular and very thin elliptical inclusions respectively, and are therefore optimal.

7. SIMULATION RESULTS

We do the numerical simulation by using a digital-image-based method (Day et al., 1992), in which we use a triangular net with central forces to represent the matrix. We place a single polygonal hole in a sample which is periodic. In order to calculate \( \alpha \) and \( v_0 \) accurately, we have to make the polygonal hole as small as possible. But on the other hand, the smaller the polygonal hole, the larger the finite size error. Therefore, there exists some tradeoff here. Furthermore, the exact area of the polygon is a rather ambiguous quantity due to the finite size effect. If the shape of the polygon is neither triangle nor hexagon, the situation is even worse, because in this case, it is harder to mimic the polygon exactly by cutting bonds in the triangular matrix. Of course, one can always increase the sample size to reduce the finite size error, but the computational time increases dramatically with the sample size. Other numerical methods, such as the finite element method, should give better results than we
were able to obtain with the finite difference method used here. Nevertheless our results are accurate enough to be interesting.

In order to get precise results, we do our simulation in two ways. One way is to calculate the area by counting the number of pixels inside the polygonal hole and each pixel corresponds to some fixed area. The other way is that we define the area by calculating the exact area enclosed by the polygon, but the strength of the spring constants for the bonds crossing the hole matrix interface are reduced by a factor of 2. This is due to the fact that each bond at this interface is shared by both the matrix and the hole. The two sets of results of $G$ and $V_0$ given by these two methods are coincident when $f$ is large, and the deviations become larger and larger when $f$ becomes smaller and smaller. Fortunately, these deviations move away in opposite directions as the hole becomes smaller and $f$ approaches 0, which enables us to take the arithmetic average of the two sets of data to reduce the finite size effect. After that, we use least-square-fit to interpolate the values of $G$ and $V_0$. Three different sample sizes, $102 \times 102$, $156 \times 156$ and $210 \times 210$, are used to estimate the finite size error and $V_0$ in (16) is chosen to be 1/3. Because of the finite size error, we can only calculate $G$ and $V_0$ up to $n = 6$ (hexagon), which is converging rapidly on the large n limit of the circle. Note that all simulations were done with the spring constants of the triangular grid $G$, $B$, and $Y$ all set equal, so that the Poisson's ratio of the host was 1/3 (Day et al., 1992). Because of the CLM theorem, this single value of $V$ is sufficient to obtain the quantities of interest $G$ and $V_0$, which are given in the Table 2.

Table 2. The values of $G$ for use in (12) and the invariant Poisson's ratio $V_0$ with estimated errors from the simulations, for regular polygonal holes.

<table>
<thead>
<tr>
<th>n</th>
<th>$G$</th>
<th>$V_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 (triangle)</td>
<td>4.23 ± 0.02</td>
<td>0.227 ± 0.003</td>
</tr>
<tr>
<td>4 (square)</td>
<td>3.39 ± 0.04</td>
<td>0.302 ± 0.006</td>
</tr>
<tr>
<td>5 (pentagon)</td>
<td>3.25 ± 0.02</td>
<td>0.325 ± 0.003</td>
</tr>
<tr>
<td>6 (hexagon)</td>
<td>3.14 ± 0.02</td>
<td>0.328 ± 0.003</td>
</tr>
<tr>
<td>0 (circle)</td>
<td>3</td>
<td>0.333</td>
</tr>
</tbody>
</table>

One special case is when the regular polygon is a square. In this case, the effective elastic moduli are not isotropic quantities. All other inclusions, with regular polygons, lead to isotropic elasticity equations (2) for the composite, and no angular averaging is required. The square is different as the equivalent directions are at right angles which is not sufficient symmetry to give isotropy, and three elastic constants, rather than two are required. We compute results for the stress parallel to the side of the square, and find that $G=3.78±0.02$ and the invariant Poisson's ratio $V_0=0.378±0.003$. Rotating the stress loading by $π/4$ so that it is along a diagonal of the square, we find that $G=3.00±0.02$ and the invariant Poisson's ratio $V_0=0.205±0.003$. Averaging $G$ and $B$ and using (15), we obtain the results shown in Table 2. This is the correct procedure to first order in the area fraction $f$.

We emphasize that the results in Table 2 are for polygonal holes in continuous materials, and the grid was just a numerical device used in the computation. The results in Table 2 show a smooth evolution from the triangle to the circle, which is the limit of a regular n-sided polygon. The shape effect is surprisingly large, especially on the invariant ratio. Note that we could have added an n=2 result for a slit, which can be obtained from the ellipse as the aspect ratio goes to zero. This would give $G=\infty$, and zero for the invariant Poisson's ratio as can be seen by taking the limits on the equations for ellipses in the next section. In Fig. 2, we show plots of the results fitted to the following curves,
\[ q = \frac{33.2}{n} \quad \text{and} \quad v_0 = \frac{1}{n} - \frac{25.8}{n^5} \]

(17)

Fig. 2  Showing the results for \( q \) and \( v \) for various \( n \)-gons, using the results from Table 2, and the power law fits (17).

These fits were made by plotting the results against various power laws and choosing the one that came closest to giving a straight line. The result (17) may be useful in extending our work beyond \( n \)=6, to estimate the difference between say an octagon and a circle. Notice that the result for the Young’s modulus for the square does not fit onto the curve, which we find surprising as we expected a monotonic progression. The square is different because of the averaging required and this may account for its different behavior.

As a concluding remark here, it is important to comment that for the geometry of polygonal holes with a finite the stress is singular in the matrix at the sharp corners. However, since it is a geometric type of singularity (Muskhelishvili, 1953; Timoshenko and Goodier, 1951) an elastic strain energy is finite (or integrable). The elastic moduli are obtained directly from the elastic strain energy.

8. EFFECTIVE MEDIUM THEORIES

The CLM theorem and the CLM transformation provide an important check on effective medium theories of composite materials in 2D. It is clearly desirable that these approximate theories should be invariant under the CLM transformation. Because the dilute result is exact, it is of course invariant under the CLM transformation. The dilute result is used as the starting point in many effective medium theories. We have tested four commonly used methods, the self-consistent method (Budiansky, 1965; Hill, 1965; Wu, 1966), the differential scheme (McLaughlin, 1977), the generalised self-consistent method (Christensen and Lo, 1979), and the Mori-Tanaka method (Mori and Tanaka, 1973; Benveniste, 1987), and we have found that in any of the above methods the CLM invariance is maintained.

Here, we will use our dilute results (12) and (16) along with Table 2 to predict the effective elastic moduli at higher volume fractions of holes by using two effective medium theories: the differential scheme and the self-consistent method.

The differential scheme is an iteration on the dilute limit such that incremental amounts of larger holes are embedded in a medium containing the previous level as an effective medium. Independent differential equations are obtained for the Young’s modulus and the Poisson’s ratio via,
\[
\frac{d\bar{E}}{E} = \frac{\partial df}{(1-f)}
\]

and

\[
dv = \left(\frac{\nu - \nu_0}{\nu - \nu_0} \right) \frac{\partial df}{(1-f)}
\]

Equations (18) and (19) can be integrated, with the initial conditions that for small \(f\), the dilute limits given in (12) and (16) are recovered. This is a standard technique that is explained in more detail in Jasiuk, Chen and Thorpe (1992). The results are

\[
\bar{E} = \frac{\nu - \nu_0}{\nu - \nu_0} (1-f)^\alpha
\]

(20)

The results (20) for the differential scheme show that the Young's modulus goes to zero, and the Poisson's ratio \(\nu = \nu_0\) as \(f \to 1\). The Young's modulus result is shown in Fig. 3 for triangles, squares and hexagons, and it gives an idea of the shape dependence of the result for a given area fraction of holes. This is qualitatively the same for all polygonal inclusions but differs in detail with the parameters \(\alpha\) and \(\nu_0\) given in Table 2. The differential scheme is set up to describe a composite containing regular polygons with a wide size variation. Equations (20) apply to the situation where there is a single type of inclusion (square etc.). However, the result can easily be generalized to an arbitrary weighting of different types of polygons (triangles, squares etc.), by taking a weighted average of \(\alpha\) and \(\nu_0\).

Fig. 3 The Young's modulus \(E^*\) using the differential scheme (20) is shown for triangles, squares and hexagons.

Similarly, we can use the self-consistent method to predict the effective elastic moduli of composites containing polygonal holes. Using the dilute result for \(E\) and \(\nu\) given in (12) and (16) we can obtain the expressions for the effective bulk and shear moduli \(K^*\) and \(G^*\) which are related to \(E\) and \(\nu\) via (3)

\[
K^* = 1 - \alpha \frac{1 - \nu}{1 - \nu} f
\]

(21)

\[
G^* = 1 - \alpha \frac{1 + \nu}{1 + \nu} f
\]

(22)
We can achieve self-consistency by replacing \( V \) by \( V^* \) on the right hand sides of equations (21) and (22).

\[
\frac{K^*}{K} = 1 - a \frac{1 - V_0}{1 - V} f
\]

\[
\frac{G^*}{G} = 1 - a \frac{1 + V_0}{1 + V} f
\]

(23)

(24)

We combine these equations via (3) to form equations for \( E^* \) and \( V^* \).

\[
\frac{E^*}{E} = \frac{V^* - V_0}{V - V_0} = 1 - df
\]

(25)

which are the same as the dilute results (12) and (16), but they hold for non-dilute concentration. The result (25) for the Poisson's ratio is plotted in Fig. 4 for the cases of triangles, squares and hexagons. It can be seen that both the invariant value of the Poisson's ratio and the critical value of the area fraction \( f \) are sensitive to the type of polygon.

![Graph showing the Poisson's ratio vs. area fraction for different polygons](image)

**Fig. 4** The Poisson's ratio \( \nu^* \), using the self-consistent scheme (25), is shown for triangles, squares and hexagons. Here, the star \( * \) marks the fixed point of the Poisson's ratio which is at \( \nu_0 \).

Substituting (25) into (23) and (24) we have the following expressions for \( K^* \) and \( G^* \)

\[
\frac{K^*}{K} = 1 - a \frac{1 - V_0}{1 - V + f g(V - V_0)} f
\]

(26)

\[
\frac{G^*}{G} = 1 - a \frac{1 + V_0}{1 + V - f g(V - V_0)} f
\]

(27)

The results (23) and (24) for the self-consistent method show that the moduli go to zero and the Poisson's ratio \( \nu^* \) at the critical area fraction \( f = 1/a \).

The present result for the self-consistent method complements an earlier study which was done for elliptical inclusions (Sen and Thorpe, 1985). For this case, the area Young's modulus and the area Poisson's ratio are
\[ \frac{E^*}{E} = 1 - \frac{f/f_c}{f} \]
\[ v^* = v - (v - v_c)f/f_c \]

where
\[ v_c = \frac{1}{f} \frac{b/a + 1}{d/a + 1} \]

and
\[ f_c = \frac{1}{f} \frac{b/a + 1}{d/a + 1} \]

Here \( b/a \) is the aspect ratio of the randomly oriented and randomly centered ellipses, and the subscript \( c \) denotes the quantity at percolation. Here, the subscript \( c \) has the same meaning as the subscript 0 used earlier in this section.

We comment that these effective medium equations at least have a structure compatible with the CLM theorem. Although this compatibility is desirable, it is of course not sufficient to ensure a good effective medium theory. A good figure of merit is given by the percolation concentration. For circular holes, equation (29) gives \( f_c = 1/3 \) which is far from the exact result of \( f_c = 0.66 \) (Xia and Thorpe, 1988). Note that in any dimension, the percolation concentration is a geometrically determined quantity that is independent of the Poisson’s ratio; any effective medium theory that is invariant under the CLM transformation ensures that this property is achieved for holes.

Finally, we comment on the results for a material with planar cracks which exist in the literature. The dilute result is
\[ \frac{E^*}{E} = 1 - \eta \]
\[ v^* = v(1 - \eta) \]

where \( \eta \) is a crack density parameter defined by \( \eta = \frac{N \alpha^2}{A} \), where \( N \) is the number of cracks, the length of the crack is \( 2a \), and \( A \) is the area of the sample. Please note that the results (12) and (16) reduce to (30) and (31) if we set \( v_0 = 0 \) and \( G_0 = \eta \).

The self-consistent result is (Gotteseanu, 1980; Benveniste, 1986)
\[ \frac{E^*}{E} = \frac{v^*}{v} = 1 - \eta \]

Note that these expressions are identical to (30) and (31). However, since the crack interaction has been taken into account here, the equations (32) are also valid in a non-dilute situation.

The differential scheme solution of Sagalnik (1973) gives
\[ \frac{E^*}{E} = \frac{v^*}{v} = e^{-\eta} \]

The Mori-Tanaka result obtained by Benveniste (1986) yields
\[ \frac{E^*}{E} = \frac{v^*}{v} = \frac{1}{1 + \eta} \]

As discussed earlier, the CLM result is particularly powerful for holes, where it shows that the Young’s modulus should be independent of the Poisson’s ratio of the material, and the Poisson’s ratio should flow to a universal value at the fixed point (Day et al., 1992). Please note that in all the results discussed in this section, the Young’s modulus is indeed independent of the Poisson’s ratio.
9. CONCLUSIONS

In this paper, we have drawn together and discussed a number of exact results for 2d composite materials. These results begin historically with the result of Michell (1899) concerning the stresses in plates containing holes, and are all brought together by a recent theorem proved by Cherkaev, Lurie and Milton (1992). We have discussed the special case of a material containing holes and showed that the effective area Young's modulus is independent of the Poisson's ratio of the matrix. We presented the results for the elastic constants of a material containing polygonal holes and also discussed other geometries such as circular, elliptical and slitlike holes (cracks). We have also pointed out that effective medium theories in 2d should reflect the transformation properties in the CCM theorem.

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REFERENCES


