The Conductivity of a Sheet Containing Inclusions with Sharp Corners

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We have written down an integral equation for the surface charge on the interface between an n-sided regular polygonal inclusion with conductivity $\sigma_i$ and a host with conductivity $\sigma_0$. The conductivity of a sheet with few inclusions can be determined from the induced dipole associated with this charge distribution. The integral equation is solved numerically and detailed results are given for $n = 3, 4$ and $6$. The circle limit is approached for large $n$. We show that our results are consistent with the reciprocity theorem for two-dimensional conducting media, and we reproduce the known analytic results in the limits when the inclusion is either a hole or a superconducting region. We also find the conductivity of a sheet containing a general $n$-pointed star. Analytic fits to our numerical results are given which are used as input for effective medium theories. The results given in this paper for the electrical conductivity, apply also to the thermal conductivity and to dielectric media.

1. Introduction

The properties of random conducting materials have been studied since the first work done by Maxwell (1973). Most of the work in the literature is concerned with spherical or ellipsoidal inclusions as these geometries can be solved in closed form. In two-dimensional (2D) systems, conformal mapping techniques (Churchill et al. 1974; Silverman 1974; Nehari 1975) can be used to obtain the complete solution when there is a single inclusion of a non-conducting material (i.e. a hole) made up of straight edges. This is done by a straightforward application of the Schwarz–Cristoffel transformation (Nehari 1974). The solution has the property that there are infinite current densities around the sharp edges of the hole. Nevertheless is has recently been shown that the conductivity of the medium can be found to first order in the area fraction $f$ of the holes, by calculating the dipole moment caused by the induced charge around the hole (Thorpe 1992). The result is

$$\frac{\sigma}{\sigma_0} = 1 - \alpha_n f + O(f^3),$$  \hspace{1cm} (1)

where $\alpha_n$ for an $n$-sided regular polygon is given by

$$\alpha_n = \frac{\tan(\pi/n)}{2\pi n} \frac{\Gamma(1/n)}{\Gamma(2/n)}.$$  \hspace{1cm} (2)

It was found (Thorpe 1992) that the conductivity $\sigma$ was perfectly well behaved as the divergent current density at the corners of the polygon is integrable.

In this paper we extend the work summarized in the previous paragraph to the general case of a two-component system where the host conductivity is $\sigma_0$ and the
inclusion conductivity is \( \sigma_1 \). This problem has been attacked before numerically for square holes by using a numerical variational technique (Milton et al. 1970). In this paper we show that the solution can be obtained in terms of an integral equation for the induced charge density on the perimeter of the inclusion. This equation can be solved numerically to arbitrary accuracy. From the induced charge density, the dipole moment and hence the conductivity can be obtained.

It is known from the reciprocity theorem (Keller 1964; Medelson 1974) that if the conductivity of a two-phase medium in 2D is \( \sigma(\sigma_0, \sigma_1) \), and if the geometry is kept fixed but \( \sigma_0 \) and \( \sigma_1 \) are interchanged, to give a conductivity \( \sigma(\sigma_1, \sigma_0) \), then these two conductivities are related by

\[
\sigma(\sigma_0, \sigma_1) \sigma(\sigma_1, \sigma_0) = \sigma_0 \sigma_1. \tag{3}
\]

A special case of (3) is when one component is present in small amounts, so that to first order in the area fraction of the inclusions, the conductivity \( \sigma \) must have the form

\[
\frac{\sigma}{\sigma_0} = 1 + \gamma G(\gamma)f, \tag{4}
\]

where

\[
\gamma = (\sigma_1 - \sigma_0)/(\sigma_1 + \sigma_0) \tag{5}
\]

and \( G(\gamma) \) is an even function of \( \gamma \). Two important limits are obtained when the inclusion is a hole (\( \gamma = -1 \)) and when the inclusion is perfectly conducting (\( \gamma = 1 \)), which we shall refer to as a superconducting inclusion. From our previous work (Thorpe 1992), we know that

\[
G(\pm 1) = a_n = \frac{\tan(\pi/n)}{2\pi n} \frac{\Gamma^4(1/n)}{\Gamma^2(2/n)}. \tag{6}
\]

The limit (6) may be regarded as the strong scattering limit. In the weak scattering limit, the problem may also be solved exactly using perturbation theory (Landau & Lifshitz 1960) to second order in \( (\sigma_1 - \sigma_0) \). The limit of weak scattering in 3D is given in Landau & Lifshitz (1960) as \( \sigma = \sigma - (\sigma - \sigma_0^2)/3\sigma \), where we use the notation in this paper and the bar represents an average. Writing a similar equation for 2D (i.e. replacing the 3 by 2), we obtain the result for small \( \gamma \)

\[
G(\gamma) = 2 + O(\gamma^2). \tag{7}
\]

It is also possible to obtain bounds on \( G(\gamma) \). The most general bounds for a two-component mixture that is macroscopically isotropic, have been given by Hashin and Shtrikman (1962) (see also Bergman 1978). The lower bound \( \sigma_1 \) and the upper bound \( \sigma_u \) of the conductivity are

\[
\sigma_1 = \sigma_0 \left[ \frac{\sigma_1 + \sigma_0 + (\sigma_1 - \sigma_0)f}{\sigma_1 + \sigma_0 - (\sigma_1 - \sigma_0)f} \right] \tag{8}
\]

and

\[
\sigma_u = \sigma_1 \left[ \frac{2\sigma_0 + (\sigma_1 - \sigma_0)(f)}{2\sigma_1 - (\sigma_1 - \sigma_0)f} \right] \tag{9}
\]

so that \( \sigma_1 \leq \sigma \leq \sigma_u \) if \( \sigma_1 \geq \sigma_0 \). These bounds are reversed if \( \sigma_1 \leq \sigma_0 \). Expanding (8) and (9) to first order in the area fraction \( f \), we obtain general bounds on \( G(\gamma) \)

\[
2 \leq G(\gamma) \leq 2/(1 - \gamma^2). \tag{10}
\]

Equations (6), (7) and (10) place strong constraints on \( G(\gamma) \) and provide useful checks upon our numerical procedures.

The conductivity of a sheet with corners

The layout of this paper is as follows. In the next section, we develop the theory that leads to the integral equation for the induced charge density around the inclusion, and show how the induced charge and hence the induced dipole moment can be obtained. In §3, we examine the singularities associated with the vertices on the interface between the two conducting regions. In §4, we discuss the numerical method of solution of the integral equation. In §5, we make a diversion to recall the exactly known result for elliptical inclusions and recast this result in the notation of this paper. In §6, we give the numerical solution for regular triangles, squares and hexagons. We develop the Taylor series and give an interpolation formula for \( G(\gamma) \). In §7 we present similar results for star-shaped inclusions and give the exact result when the star becomes a hole by using conformal mapping. Finally in §8 we use these results to develop an effective medium theory for the case when the inclusions are no longer dilute.

2. Derivation of the integral equation

Since the electric field is discontinuous at the boundary of the conducting inclusion, an induced charge develops at the boundary. We derive an integral equation for this charge density. The term charge density will be used for the charge per unit length in the \( z \) direction of the hypothetical cylindrical extension of the boundary into the third dimension.

In 2D electrostatics the electric potential \( \phi \) at a point \( r \) outside a closed charged boundary in the presence of an applied constant field \( E_\infty \) in the \( x \) direction is

\[
\phi(r) = -x E_\infty - \frac{1}{2\pi} \oint \ln |r - r'(s')| \rho(s') \, ds',
\tag{11}
\]

where \( r(s) \) is the vector position of a point in the boundary measured by the coordinate \( s \), \( \rho(s) \) is the induced charge density, and \( s \) is defined as the distance along the boundary from some arbitrary starting point.

The discontinuity of the electric field at the boundary between two conductivities is characterized by

\[
\sigma_0 \mathbf{n} \cdot \mathbf{E}_0 - \sigma_1 \mathbf{n} \cdot \mathbf{E}_1 = 0,
\tag{12}
\]

where \( \sigma_0 \) and \( \sigma_1 \) are the outside and inside conductivities respectively, and \( \mathbf{n} \) is the outward normal. This discontinuity also represents a surface charge so that

\[
\mathbf{n} \cdot \mathbf{E}_0 - \mathbf{n} \cdot \mathbf{E}_1 = \rho,
\tag{13}
\]

where \( \rho \) has the same meaning as in (11). Combining (12) and (13) one obtains

\[
\mathbf{n} \cdot \mathbf{E}_0 (1 - \sigma_0/\sigma_1) = \rho.
\tag{14}
\]

But since \( \mathbf{E}_0 = -\nabla \phi \) we may combine (11) and (14) to obtain

\[
\rho(r) = \beta \mathbf{n} \cdot \mathbf{E}_\infty + \frac{\beta}{2\pi} \oint \mathbf{n} \cdot \nabla \ln |r - r'(s')| \rho(s') \, ds',
\tag{15}
\]

where \( r \) is just outside the boundary \( r(s) \) and where \( \beta = (1 - \sigma_0/\sigma_1) \). Because of the singular nature of the kernel in (15) it may be broken into a \( \delta \)-function part and a principal value part and one obtains

\[
\rho(s) = \beta \mathbf{n}(s) \cdot \mathbf{E}_\infty + \frac{1}{2} \beta \rho(s) + \frac{\beta}{2\pi} \oint \mathbf{n}(s) \cdot \nabla \ln |r(s) - r'(s')| \rho(s') \, ds'.
\tag{16}
\]
Notice that since \( r(s) \) is now exactly on the line together with \( r'(s') \), the kernel in (16) is well behaved (Lovitt 1950; Pogorzelski 1966) at the point \( s = s' \).

Collecting terms in (16), carrying out the differentiation, and noticing that

\[
n(s) = (dy/ds, -dx/ds)
\]

we have the final form of the integral equation

\[
\rho(s) = 2\gamma \frac{dy}{ds} E_\gamma + \gamma \int \frac{dy}{ds} \left( \frac{dy/ds(x(s) - x(s'))}{(x(s) - x(s'))^2 + (y(s) - y(s'))^2} \right) \rho(s') \, ds'.
\]

Finally we find the dipole moment of this charge distribution which is given by

\[
\mu = \int x(s) \, \rho(s) \, ds.
\]

The dipole moment \( \mu \) can be shown to be an odd function of \( \gamma \) so that the quantity \( G(\gamma) = \mu/(\gamma E_\gamma) \) is an even function of \( \gamma \) and as can be seen from (19) it has value 2 when \( \gamma = 0 \) in agreement with the result (7).

The analytic structure in the complex \( \gamma \) plane is complicated by the presence of discontinuities in the kernel at the corners of the polygonal figures. If the boundary is smooth so that the kernel is continuous then Fredholm theory (Lovitt 1950) shows that \( G(\gamma) \) is meromorphic (i.e. analytic except for poles and a possible singularity at infinity). An example of this behaviour is the ellipse discussed in §5. Any discrete approximation of the kernels for polygons leads to a set of simultaneous equations and thus a finite number of eigenvalues. This leads in turn to a finite number of poles in the complex \( \gamma \) plane. We believe, however, that because of the large number of such poles not at infinity, that they may lie along branch cuts of the exact solution. The poles we find lie along the positive and negative real axes starting at points outside the range \((-1, 1)\). The closeness of the approximation discussed in §6 gives some support to this idea, although some of the poles found numerically lie closer to the origin than the branch cuts of the approximation (42).

3. Singular nature of the charge density

At the corners between adjacent polygonal sides, one expects in general that the charge density will be singular. This result is familiar in the case of non-conducting holes or conducting points extending into a non-conducting medium (Jackson 1975). In the case of two different conductivities, the charge density also has a singular behaviour as is shown by the following argument. The kernel of the integral equation (18) is not finite at the corners. As mentioned previously this violates the usual restrictions on the proof of Fredholm theory (Lovitt 1950). If the charge density approaches infinity near the corners then, since this infinity is not in the inhomogeneous term, the infinite behaviour must be replicated by the kernel. We assume that the charge density is proportional to a negative fractional power of the distance from the corner and show that this behaviour is replicated by application of the kernel and that the power depends on the angle at the corner and also on the conductivity ratio parameter \( \gamma \).

Near a corner the kernel can be written in terms of the distances from the corner by making the substitutions \( s = e_0 + \epsilon \) and \( s' = e_0 - \epsilon' \) in (18), where \( e_0 \) is the corner position. We seek replication of a singular region by application of the kernel. The operation of the kernel on a singularity on one side of a corner will produce a
The conductivity of a sheet with corners

Figure 1. The exponent $\nu$ from (21) determines the divergence of the charge density as the corners of the inclusion are approached and is shown for various regular $n$-sided regular polygons as a function of the conductivity ratio parameter $\gamma$.

singularity on the other side of the corner. Replication will occur after two steps provided that after one step the singular power is replicated with a multiplicative factor $\pm 1$.

$$
\varepsilon^{-\nu} = \pm \frac{\gamma}{\pi} \int_0^\infty \frac{\varepsilon' e^{-\nu} \sin \theta}{\varepsilon^2 + 2\varepsilon' \cos (\pi - \theta) + \varepsilon'^2} d\varepsilon',
$$

(20)

where $\theta$ is the interior angle of the corner ($0 \leq \theta \leq \pi$). Here the upper limit is taken to be infinity because for $0 < \nu < 1$ the integral converges well for large $\varepsilon'$ and we do not expect that limit to affect the singularity. The integral in (20) can be evaluated in closed form and we find that $\varepsilon^{-\nu}$ factors on the right so that (20) is solved provided

$$
1 = \pm \gamma \sin \left[(1 - \nu)(\pi - \theta)/\sin (\pi \nu)\right].
$$

(21)

For $0 < \nu < 1$ both sines are in the first two quadrants so that for $\gamma = -1$ and taking the minus sign in (21), we obtain the familiar result (see Jackson 1975)

$$
\nu = 1 - \pi/(2\pi - \theta).
$$

(22)

For two different conductivities (i.e., for $\gamma \neq 1$) we find that the power $\nu$ depends on the conductivity ratio parameter $\gamma$. In figure 1, we show the most divergent root from (21) as a function of $\gamma$ for various regular polygons where $(\pi - \theta) = 2\pi/n$. The result (21) is particularly simple for the square, where $\theta = \pi$ and we find

$$
\nu = (2/\pi) \arcsin (\pi |\gamma|).
$$

(23)

The various limiting behaviours in figure 1 are as follows. For small $\gamma$, we have

$$
\nu = (|\gamma|/\pi) \sin (2\pi/n)
$$

(24)

and when $\gamma = \pm 1$, we find that

$$
\nu = 2/(n + 2).
$$

(25)

The exponent $\nu$ cannot exceed 1 as this would imply an infinite charge (as opposed

to an infinite charge density) and it can be seen from figure 1 that this value is reached when $\gamma = \frac{1}{2}$. Equation (21) is singular at the point where $\nu = 1$ and we speculate that the corresponding value of $\gamma = \frac{1}{2} \nu$ marks the beginning of branch cut(s) in the complex $\gamma$ plane. Indeed these values of $\gamma$ agree fairly well with the ends of rows of poles found numerically. Our calculated charge densities have been compared with the singular values predicted by (21) and here the agreement is precise.

The result (21) can also be obtained from elementary electrostatics. Aligning the symmetry plane of the vertex with the $x$-axis, and putting the positive $x$-axis in the center of the region with conductivity $\sigma_1$ defined by the angle $\theta$ where $(0 \leq \theta \leq \pi)$, we write the potential $V$ in this region as

$$V = A r^{1-\nu} \cos [(1-\nu) \phi] ,$$

(26)

where $(r, \phi)$ are the usual polar coordinates. A similar solution of Laplace's equation holds in the remaining region around the vertex where the conductivity is $\sigma_0$

$$V = B r^{1-\nu} \cos [(1-\nu) (\pi - \phi)].$$

(27)

The coefficients $A, B$ can be eliminated by using continuity of the potential and current flow across the interfaces between the two solutions (26) and (27). The result is (21) with the plus sign. The minus sign in (21) is obtained by writing down solutions like (26) and (27) with the cosines replaced by sines. This derivation of (21) has the advantage that it is also valid when $\nu$ is negative. Symmetry at a vertex can prevent the occurrence of the most singular $\nu > 0$ solution and then the next most singular value will prevail, giving a negative $\nu$ solution. When the negative sign is chosen in (20) and (21), this implies that the signs of the charges on the two sides of the vertex are opposite. For the positive sign in (20) and (21), $\nu$ has the same sign as $\gamma$. Thus if $\gamma > 0$ the divergent charge has the same sign on the two sides of the vertex. If $\gamma < 0$ the divergent charge has opposite signs on the two sides of the vertex. However, in either case if symmetry requires the opposite behaviour, then the non-divergent negative value of $\nu$ will dominate.

4. Numerical method

Discretizing the integral in (18) by using gaussian points and weights on each straight line segment, allows this integral equation to be solved by matrix methods. However, significant improvement in convergence is achieved by mapping $s$ in such a way that the singularities at the endpoints of the segments are not infinite and by expanding $\rho$ in Lagrange interpolating polynomials (Abramowitz & Stegun 1965) and integrating these polynomials carefully with the kernel.

The numerical analysis has to take account of the fact that the boundary consists of straight line segments. Therefore we begin by breaking the integration into segments

$$\rho_n(s) = \rho_n^0(s) + \gamma \sum_{m} \int_{s_{m-1}}^{s_{m}} K_{nm}(s,s') \rho_n(s') ds' ,$$

where $s$ now measures the length along the appropriate line segment and $K$ is the kernel

$$K_{nm}(s,s') = \frac{n_n^{-1} [r_n(s) - r'n_m(s')]}{\pi |r_n(s) - r'n_m(s')|^2} .$$

(29)

*Proc. R. Soc. Lond. A (1992)*
Note that $K_{nm}$ is zero if $n = m$. Numerical problems arise because $\rho_n(s)$ is singular at the end points of the interval $s \in (0, L_n)$. We therefore make a transformation from the variable $s$ to the variable $t$ of the form

$$s = L_n f(t),$$

(30)

where $f(0) = f(1) = 0$ and several derivatives of $f$ are zero at the end points. Specifically, we derived $f$ from the formula

$$\frac{df}{dt} = 140 t^3 (1 - t)^3$$

(31)

so that

$$f = 35 t^4 - 84 t^5 + 70 t^6 - 20 t^7.$$  

(32)

This transformation was chosen so that the functions $\chi_n(t)$ below would go to zero as strongly at $t$ at the end points if the original singularities are no worse than $s^{-1}$, which is the worst case for the usual physical region (i.e. $\theta = 0$ and $\gamma = \pm 1$, see §3). Substituting (30) into (28) and multiplying (28) by $df/dt$ yields the following set of integral equations

$$\chi_n(t) = \chi_n^0(t) + \gamma \sum_m \int_0^1 H_{nm}(t, t') \chi_m(t') \, dt',$$

(33)

where $\chi$ (and similarly $\chi^0$) is defined

$$\chi_n(t) = L_n \frac{df}{dt} \rho(L_n f(t))$$

(34)

and where $H$ is defined

$$H_{nm}(t, t') = L_n \frac{df}{dt} K_{nm}(L_n f(t), L_n f(t')).$$

(35)

Even after this transformation, it is still not good to approximate the integrals with an integration rule because the end points of the kernel have strong behaviour. It is best to expand $\chi_n(t)$ in Lagrange interpolating polynomials (Abramowitz & Stegun 1965)

$$\chi_n(t) = P_i(t) \chi_n(t_i).$$

(36)

With this expansion, the integral can be calculated carefully down to the end points and the integral equation is now reduced to a set of simultaneous equations

$$\chi_n(t_i) = \chi_n^0(t_i) + \gamma \sum_{m_j} H_{ni, m_j} \chi_m(t_j),$$

(37)

where the matrix $H$ is given by

$$H_{ni, m_j} = \int_0^1 H_{nm}(t_i, t') P_j(t') \, dt'.$$

(38)

These integrals were calculated with considerable precision. This integration was the predominant computer time constraint since one integration has to be done for each element of the final matrix. Finally with similar substitutions in (19) the moment $\mu$ can be calculated

$$\mu = \sum_{ni} \left[ \int_0^1 x(L_n f(t)) P_i(t) \, dt \right] \chi_n(t_i).$$

(39)

With these techniques we were able to obtain confidently 5 or 6 digits with only 16 points in each of the intervals.

*Proc. R. Soc. Lond. A (1992)*
5. Elliptical inclusions

A useful example that can be solved exactly is provided by the case of elliptical inclusions. If the semi-major and semi-minor axes of the ellipse are \( a \) and \( b \) respectively, then the conductivity parallel to the \( a \)-axis is given by (Landau & Lifshitz 1960)

\[
\sigma / \sigma_0 = 1 + \left( (\sigma_1 - \sigma_0) / a \sigma_0 + b \sigma_0 \right) / (a + b) f. \tag{40}
\]

Averaging over the two orientations of the elliptical inclusions, which is equivalent to an isotropic average, we obtain the conductivity in the form (4) with

\[
G(\gamma) = 2 / \left[ 1 - \left( (a - b) / (a + b) \right)^2 \gamma^2 \right]. \tag{41}
\]

This result is instructive in the context of this paper because of the analytic structure of \( G(\gamma) \) which contains two simple poles on the real \( \gamma \) axis. These poles are located at \( \pm (a + b) / (a - b) \) with residues \( \pm (a + b) / (a - b) \). In the needle limit, these poles are at \( \gamma = \pm 1 \) and they move out to infinity as the circle limit is approached. We note that the result (41) lies between the bounds for \( G(\gamma) \) given in (10) and that the lower bound is realised by the circle and the upper bound is realised in the needle limit. The singularities of \( G(\gamma) \) are restricted to the real axis in all cases (Bergman 1978).

6. Numerical solution for regular polygons

Figure 2 shows \( G(\gamma) \) for the triangle \( (n = 3) \), square \( (n = 4) \) and hexagon \( (n = 6) \). The result is not dependent on the angle of presentation of these regular figures to the incoming current, as required by symmetry. The points in figure 2 are the result of the solution of the integral equation (18) while the curves passing through the points show fits by polynomials. Table 1 gives these polynomials for the triangle, square and hexagon. It is found that these fits agree with the calculated points to four decimal places respectively in \( G(\gamma) \). Also shown in figure 2 are parabolas adjusted to the end point, \( \gamma = 1 \). These parabolas are clearly distinct from the correct results and show that the present results represent a considerable improvement on the earlier results of Milton et al. (1970).

The polynomials in table 1 are from the Neuman series for (18) and therefore represent the Taylor series for \( G(\gamma) \) about the origin. Because all of the coefficients are positive, the magnitude of the function on the unit circle is bounded by the magnitude at \( \gamma = 1 \). In fact the nearest singularity must be on the real axis. For these reasons it is expected that these approximations will be as valid anywhere inside the unit circle as on the real axis between \( -1 \) and \( 1 \).

We find that our numerical results can be well fit with the approximate formula

\[
G(\gamma) = 2 / \sqrt{1 - \gamma^2}, \tag{42}
\]

where the constant \( c \) is chosen to reproduce the exact result at \( \gamma = \pm 1 \) which from (6) leads to

\[
c^2 = 1 - \left[ \frac{2}{\alpha_n} \right]^2 = 1 - \left[ \frac{4\pi n}{\tan (\pi/n)} \Gamma^2(2/n) / \Gamma^2(1/n) \right]^2. \tag{43}
\]

This is shown by the plot in figure 3 which compares this formula with our numerical results. The formula (42) has branch points at \( \gamma = \pm 1 / c \). To judge how good a fit (42) is, we consider the next term and write as

\[
G(\gamma) = \left[ 2 / \sqrt{1 - \gamma^2} \right] \left[ 1 + a \gamma^2 (1 - \gamma^2) + \ldots \right]. \tag{44}
\]
Figure 2. The quantity $G(\gamma)$ for various regular $n$-sided polygons. The diamonds are computed points and the solid curves passing through them are evaluations of Taylor series given in table 1. For the triangle and square, comparison parabolas are adjusted to fit the end points as shown by the dashed lines.

Figure 3. $2/G(\gamma^2)$ against $\gamma^2$ for the triangle, square and hexagon. The straight line result (42) is a very good approximation to the numerical results shown by diamonds. Although the approximation is striking it is not exact as can be seen by very careful examination of the line for the triangle. The crosses at $\gamma^2 = 1$ are exact results.

<table>
<thead>
<tr>
<th>Polygon</th>
<th>Taylor Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>triangle</td>
<td>$G(\gamma) = 2 + 0.40855\gamma^2 + 0.11564\gamma^4 + 0.03796\gamma^6 + 0.01266\gamma^8 + 0.00452\gamma^{10}$ + $\ldots$</td>
</tr>
<tr>
<td>square</td>
<td>$G(\gamma) = 2 + 0.16159\gamma^2 + 0.02233\gamma^4 + 0.00369\gamma^6$ + $\ldots$</td>
</tr>
<tr>
<td>hexagon</td>
<td>$G(\gamma) = 2 + 0.04602\gamma^2 + 0.00239\gamma^4 + 0.00016\gamma^6 + 0.00001\gamma^8 + \ldots$</td>
</tr>
</tbody>
</table>

*Table 1. Taylor series for polygonal inclusions of conductivity ratio parameter $\gamma$.*

*Proc. R. Soc. Lond. A (1992)*
and from table 1, we find that indeed $a$ is very small and equal to $0.0045$, $-0.0016$ and $-0.0004$ for $n = 3$, 4 and 6 respectively. Thus (42) provides a fit good to $10^{-2}$ for all $n$ for $\gamma^2 \ll 1$. If a more accurate fit is required, the Taylor series in table 1 can be used which is good to $10^{-4}$.

It was suggested in §3, on the basis of the divergent charge associated with the corners of the inclusion, that the branch cuts must begin at $\pm \gamma = \frac{1}{2n}$ for regular $n$-sided polygons. This leads to $\pm \gamma = 1.5$, 2 and 3 for $n = 3$, 4 and 6 respectively, compared with $\pm \gamma = 1/c = 1.58$, 2.46 and 4.62 from the approximate analytic form (42).

7. Star-shaped inclusions

In figure 4, we show an ($n = 3$)-pointed star-shaped inclusion. In the limit when $\gamma = \pm 1$, and the inclusion is either a hole or a superconducting inclusion, we can use conformal mapping techniques (Thorpe 1982) to find the conductivity $\sigma$ to first order in the area fraction $f$. We map the star-shaped area in the $x$-plane onto a unit circle in the $w$-plane. The $n$ points around the inner circle of radius $b$ in figure 4 and the $n$ points around the outer circle of radius $a$, map onto $2n$ points uniformly spaced around the unit circle in the $w$-plane. This conformal mapping may be written

$$\frac{dz}{dw} = \left(\frac{w^n - 1}{w^n}\right)^{1-2\theta/n} \left(\frac{w^n}{w^n + 1}\right)^{1-2\gamma^2/n}$$

(45)

where

$$\tan \tau = \frac{b \sin \left(\frac{\pi}{n}\right)}{a - b \cos \left(\frac{\pi}{n}\right)}.$$

(46)

The acute angles at the outer tips of the star have an angle $2\theta$, the turning angle at the outer points on the large circle is $\pi - 2\theta$, and the turning angle at the inner points on the small circle is $\pi - 2(\theta - \pi/n)$. These are used in the Schwarz-Cristoffel transformation (Churchill et al. 1974) to obtain the mapping (45). By writing down the distance between adjacent points in the star and the corresponding points on the unit circle, we find that

$$b \sin \left(\frac{\pi}{n}\right) = 2^\gamma n \pi \Gamma \left(\frac{1}{n} + \frac{\theta}{\pi}\right) \Gamma \left(\frac{1}{n}\right) \Gamma \left(\frac{\theta}{\pi}\right).$$

(47)

The area of the star is

$$nb^2 \sin \left(\frac{\pi}{n}\right) \sin \left(\frac{\pi}{n} + \theta\right) / \sin \theta$$

(48)

and the conductivity is then obtained in the form (1), where

$$\alpha_n = 2 \left(\frac{\text{area circle}}{\text{area star}}\right) = \frac{2 \sin \left(\frac{\pi}{n}\right) \sin \theta}{2^\gamma n \pi \sin \left(\frac{\pi}{n} + \theta\right)}$$

(49)

and the factor 2 in (49) comes from the circle result in the $w$-plane.

The formula (49) reduces to (2) when we put $\theta = \frac{1}{2} \pi - \pi/n$. We also obtain the circle limit $\alpha = 2$ when $n \to \infty$. The result (49) can be written in a very compact form in terms of the number of inclusions per unit area $N = f / \text{area star}$ to give

$$\frac{\sigma}{\sigma_0} = 1 + 2\pi N \left[ \frac{a \Gamma(1 - 1/n - \theta/\pi)}{2^\gamma n \pi \Gamma(1 - 1/n) \Gamma(1 - \theta/\pi)} \right]^2.$$

(50)

Figure 4. The geometry of an \(n = 3\)-pointed star. The shape is determined by the ratio of the radii of the inner circle \(b\) to outer circle \(a\).

Figure 5. The quantity \(G(\gamma)\) for an \(n = 3\)-pointed star or tesselated triangle. The radius of the inner circle is \(b\) and of the outer circle \(a\) and we show results for \(b/a = 1\) (hexagon), \(b/a = \frac{1}{2}\) (equilateral triangle), \(b/a = \frac{4}{5}\) and \(b/a = \frac{1}{5}\). The diamonds are computed and the dashed lines are for the guidance of the eye. The crosses are exact results for \(\gamma = \pm 1\) which are coincident with the computed results.

When \(\theta = 0\), we obtain the stick limit, which corresponds to \(n\) sticks pointing outwards radially from a central point

\[
\sigma/\sigma_0 = 1 - 2n a^2 N/2^{2/n}.
\]

(51)

When \(n = 2\), this result reduces to the well-known result

\[
\sigma/\sigma_0 = 1 - \frac{1}{2} n a^2 N
\]

(52)

for a dilute collection of randomly oriented sticks, each of length \(2a\) (Thorpe 1992). This result can also be obtained from the general result for an ellipse (41) as the aspect ratio tends to zero (Xia & Thorpe 1988; Tobochnik et al. 1989).

The star geometry for an inclusion of conductivity ratio parameter \(\gamma\) can be solved numerically by using the technique described in §2. In figure 5, we show the results for \(G(\gamma)\) for a series of tesselated triangles \((n = 3)\). For \(b/a = 1\), we find that \(\theta = 60^\circ\).
and we recover the result for the hexagon, \( \alpha_n = 2.0486 \ldots \); for \( b/a = \frac{1}{2} \), we find that \( \theta = 30^\circ \) and we recover the result for the triangle \( \alpha_n = 2.5811 \ldots \); for \( b/a = \frac{1}{4} \), we find that \( \theta \approx 14^\circ \) and we obtain the new result \( \alpha_n = 4.3446 \ldots \); and for \( b/a = \frac{1}{6} \), we find that \( \theta \approx 7^\circ \) and we obtain the new result \( \alpha_n = 8.1219 \ldots \). In figure 5, these results are shown by the crosses at the endpoints \( \gamma = 1 \). We note that as the star becomes more pointed the upper limit of the allowed range for \( G(\gamma) \) given in (10) is approached.

8. Effective medium theory

The results of this paper are for the dilute limit when the number of inclusions is small. These results can be extended approximately to higher concentrations by using them as the input for an effective medium theory. If inclusions with conductivity \( \sigma_i \) are present with area fraction \( f_i \) in a matrix with conductivity \( \sigma_m \), then the conductivity \( \sigma \) of the material is given by

\[
\sigma = \sigma_m + 2\sigma_m \sum_i f_i \frac{\sigma_i - \sigma_m}{\sigma_i + \sigma_m} G\left(\frac{\sigma_i - \sigma_m}{\sigma_i + \sigma_m}\right).
\]

We can achieve self-consistency by putting \( \sigma = \sigma_m \) and obtain

\[
\sum_i f_i \left(\frac{\sigma_i - \sigma}{\sigma_i + \sigma}\right) G\left(\frac{\sigma_i - \sigma}{\sigma_i + \sigma}\right) = 0,
\]

where \( \Sigma_i f_i = 1 \). This equation can be solved for a given shape of inclusion by inserting the appropriate analytic approximation for \( G(\gamma) \) and solving (54) numerically. Of particular interest is the case of holes with area fraction \( f \), in a host with conductivity \( \sigma_h \) where (54) reduces to

\[
\left(\frac{\sigma_h - \sigma}{\sigma_h + \sigma}\right) G\left(\frac{\sigma_h - \sigma}{\sigma_h + \sigma}\right) = G(1) \frac{f}{1-f},
\]

where for example \( G(1) \) is given by (6) for polygonal holes and by (49) for stars. In the dilute limit we recover the result (4) and as percolation is reached, \( \sigma \to 0 \) and we obtain the critical area fraction \( f_c = \frac{1}{3} \) for all shapes of star. Using the analytic approximation (42) in (55), we can write the result as

\[
\sigma/\sigma_h = 1 - 2f/[1 + \sqrt{1 + (1 - 2f)(2/\alpha f)^2}]\]

The main difference between the result (56) for stars and the result for circles is that (56) becomes linear for circles but is always sublinear for stars and polygons. We show the result for triangles, squares, hexagons and circles in figure 6. In all cases this effective medium theory gives \( f_c = \frac{1}{3} \) which is known to be quite inaccurate for circular holes with random centres, where overlap is allowed, and percolation occurs at \( f_c \approx 0.34 \) (Xia & Thorpe 1992). However, the above approach may be a useful indicator of the kind of effect that shapes other than ellipses can produce for a range of \( f \) beyond the dilute limit but away from percolation.

The results in this section can also be applied to pure dielectric or to pure conducting media where all the quantities are real or provided \( |\mu| \leq 1 \). The results obtained in this paper for the electrical conductivity apply also to the thermal conductivity and to the dielectric function if we replace \( \sigma \) everywhere with \( \kappa \). This is because the conductivity is proportional to the imaginary part of the dielectric constant. We have chosen to use the language of conductivity in this paper as the
flow patterns, etc., are easier to visualize. For complex dielectric constants the Taylor series in table 1 should be good inside the unit circle and have a circle of convergence of about $|\gamma| \leq \frac{1}{2}a$. Our form (42) for $G(\gamma)$ should not be trusted blindly in the entire complex plane but is probably reasonable as long as the singularities are not approached too closely.

9. Conclusions

In this paper we have shown how the conductivity can be calculated for a general inclusion with straight sides and points. Even though the induced charge density is divergent at the corners of the inclusion, the conductivity of the sheet is well behaved and can be obtained from the induced dipole moment. The numerical method developed here presents no difficulty when the boundary is not convex, as shown by the calculation of the star. This approach could be extended to sheets containing many inclusions.

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References


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