THE ELASTIC MODULI OF A SHEET CONTAINING CIRCULAR HOLES

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ABSTRACT

We apply computer simulation techniques to obtain the elastic moduli of a matrix containing circular holes. As the area fraction of holes increases, the Young's modulus of the composite decreases from $E_0$, until it eventually vanishes at the percolation threshold. We study three distinct geometries: (a) periodically centered circular holes on a honeycomb lattice, (b) periodically centered circular holes on a triangular lattice, and (c) randomly centered circular holes. All three cases have the same dilute limit that can be calculated exactly. By examining the narrow necks between adjacent circles, we have calculated the critical behavior for the regular cases and obtain critical exponents of 1/2 or 3/2, depending on the local breakdown mode at the necks. For (c) we compare our results with an effective-medium theory, which predicts that the Poisson's ratio tends to 1/3 as the percolation threshold is approached. Independent of its value in the pure system, our results are also compared with recent experimental results. Based on this work, we propose that the relative Young's modulus $E/E_0$ of a two-dimensional sheet containing circular holes, overlapping or not, is the same for all materials, independent of the Poisson's ratio $\nu_0$, for any prescribed geometry.

INTRODUCTION

Calculating the elastic properties of a material that is made up of randomly distributed phases that have different elastic properties is a difficult problem that has received much attention (WATTS et al., 1976). Materials of interest range from semiconductor alloys, where the random mixing of elastic properties is on atomic length scales, to fiber-reinforced materials, like SiC-whisker-reinforced ceramics or steel-fiber-reinforced concrete, where the mixing occurs on macroscopic length scales. Atomic-scale composite problems are treated with lattice calculations (FENG et al.,

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1985) while macroscopic problems are usually treated with continuum mechanics (Christensen, 1979).

This paper describes a merging of these two methods, using a new, discretized-spring scheme realized on a digital-image-based model of a two-dimensional (2d) two-phase material, where one of the phases has zero elastic moduli. This second phase is generated by punching out circular holes in a sheet. In this paper, we study circular holes of equal size in three distinct geometries. The three cases studied are: (a) periodically centered circular holes on a honeycomb lattice, (b) periodically centered circular holes on a triangular lattice, and (c) randomly centered circular holes. In the first two cases, there is no overlap between circles and we increase the radii of the circles until they touch at percolation. In case (c), there is no restriction on the circles overlapping as the holes are randomly centered. The remaining area fraction of the matrix is denoted by $p$. In (a) and (b), the matrix area fraction remaining is given by

$$p = 1 - na,$$

(1)

where $n$ is the number of holes per unit sheet area, each having area $a$. In (c), the matrix area fraction is given by (Roach, 1968; Xia and Thorpe, 1988)

$$p = e^{-na}.$$  

(2)

If the exponential in (2) is expanded, then the term in $n^r$ corrects for the mutual overlap of $r$ circles. The simple exponential form is obtained because of the complete randomness. In this initial study, we have avoided the interesting case of random circles that cannot overlap, as that geometry is somewhat more complex to generate and describe.

The layout of this paper is as follows. In the next section, we describe how the grid is set up and then in the following section we develop the numerical algorithm. Before discussing the results, we devote two sections to a discussion of the breakdown mechanism at the necks between adjacent circles, and to the results in the dilute limit. The results in both these sections are exact and serve as useful checks on our numerical algorithm. We compare our results with an effective-medium theory (EMT) for the random case (c). We then devote a section to each of the three geometries [(a)–(c)]. Finally, we compare the results for (c) with some recent experiments.

**DIGITAL-IMAGE-BASED MODEL**

Representing images with pixels on a lattice is a technique that has been used for many years for quantitative image analysis (Castelman, 1979). Usually a square lattice of equal-sized square pixels is superimposed over an image, with the pixels being assigned to different phases of the picture according to whether or not the center of the pixel falls inside a phase boundary. Once the original continuum picture is thus represented digitally, all mathematical operations of interest can be carried out on the underlying lattice, greatly simplifying their application.

The idea of breaking up a continuum into a lattice, and then carrying out mathematical operations on the lattice instead of the continuum, is of course the basic idea behind finite-difference (Press et al., 1989) and finite-element methods (Cook et al.,
for solving partial differential equations, and has been used for many years. The approach is to start out with continuum partial differential equations, and convert them into linear algebraic equations by approximating derivatives by quotients of differences. This can be done for the partial differential equations of elasticity as well as for any other set of equations.

In this paper, we take a slightly different approach. We represent a continuum model by pixels, and then use the underlying pixel lattice and material properties to define a spring network having force constants that model the elastic properties of the original continuum material. The simulations were done on a rhombahedral section of a lattice of hexagonal pixels with \( L \) pixels on each edge, as shown in Fig. 1(a)–(c). We use periodic boundary conditions so that the \( L \times L \) section is the unit supercell of an infinite periodic lattice. For geometry (a), we have two holes per unit supercell placed as shown in Fig. 1(a). Because of the periodic boundary conditions, this forms a periodic array of holes centered on the sites of a honeycomb superlattice. The diameter of the holes can range up to \( L/\sqrt{3} \) when the holes touch and percolation occurs. For geometry (b), we have one circular hole at the center of the unit supercell as shown in Fig. 1(b). Because of the periodic boundary conditions, this forms a periodic array of holes centered on a triangular superlattice with lattice constant \( L \). The diameter of the holes can range up to \( L \) when the holes are close packed and percolation occurs. For geometry (c), many circles of fixed diameter \( D \) are placed at random in the supercell. The circles are all centered on pixels so that their digitized shapes are all exactly the same. In our simulations we chose \( L = 210 \) for all three geometries although some larger lattices were studied to check the results. For geometry (c), the diameter of the circle was \( D = 11 \). Obviously in geometries (a) and (b) we have a much better approximation to a continuum than is possible in geometry (c). In all three cases, the elastic network is defined by connecting nearest-neighbor pixels with central force springs of the appropriate force constant, as discussed in the next section.

A space-filling triangular network of hexagonal pixels is used instead of a square network of square pixels, because the square lattice is unstable with only nearest-neighbor central-force springs (Feng and Sen, 1984; Feng et al., 1985).

**Numerical Method**

Once an image is defined, as in Fig. 1(a)–(c) with a certain number of holes, a spring network for the matrix is created by connecting each pixel to its six nearest neighbors with springs of the appropriate force constant. Since the perfect triangular network with all the same nearest-neighbor springs obeys Cauchy’s relation and is elastically isotropic, it is necessary to use more than one kind of spring in order to have a variable Poisson’s ratio (Feng et al., 1985). Figure 2 shows how three different kinds of springs, \( \alpha \), \( \beta \) and \( \gamma \), are defined in the triangular spring lattice, so as to preserve isotropy, but allow for different values of the Poisson’s ratio in the perfect lattice before any holes are introduced. There are three different kinds of sites, each of which forms its own triangular lattice, expanded by a factor \( \sqrt{3} \) over the elementary triangular grid shown in Fig. 2. Each of the elementary triangles in Fig. 2 has an \( \alpha \)-, \( \beta \)-
Fig. 1. (a) Unit rhomboidal supercell with 42 hexagonal pixels on each edge. There are two circular holes in each supercell, positioned as shown. The periodic boundary conditions make this equivalent to an infinite array of holes centered on a hexagonal superlattice. In this geometry the porosity is increased by increasing the diameter of the holes. In the actual simulations, the area of the supercell was 25 times larger than shown with 210 pixels on each edge. (b) Unit rhomboidal supercell with 42 hexagonal pixels on each edge. There is one circular hole in each supercell, positioned as shown. The periodic boundary conditions make this equivalent to an infinite array of holes centered on a triangular superlattice. In this geometry the porosity is increased by increasing the diameter of the holes. In the actual simulations, the area of the supercell was 25 times larger than shown with 210 pixels on each edge. (c) Unit rhomboidal supercell with 42 hexagonal pixels on each edge and seven four randomly centered 11 pixel diameter circular holes. The superlattice has no particular symmetry. In this geometry the porosity is increased by increasing the number of holes, while the diameter of the holes is kept fixed. In the actual simulations, the area of the supercell was 25 times larger than shown with 210 pixels on each edge.
and \( \gamma \)-bond for an edge. The \( \alpha \)-, \( \beta \)- and \( \gamma \)-bonds each form their own honeycomb lattices which interpenetrate one another. For the perfect lattice, the area bulk modulus \( K_0 \) and the shear modulus \( \mu_0 \) are given by

\[
K_0 = \frac{1}{\sqrt{12}} (\alpha + \beta + \gamma),
\]

\[
\mu_0 = \frac{27}{16} \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right)^{-1}.
\]

The perfect lattice is isotropic because the \( \alpha \)-, \( \beta \)- and \( \gamma \)-springs each form three interlocking honeycomb lattices, which are elastically isotropic. In two dimensions, the Young’s modulus \( E \) and the area Poisson’s ratio \( v \) can be found from the area bulk modulus and the shear modulus \( \mu \) through (Thorpe and Sen, 1985)

\[
E = \frac{4K\mu}{K + \mu},
\]

\[
v = \frac{K - \mu}{K + \mu}.
\]

For the perfect lattice, using (3)–(6), the Young’s modulus \( E_0 \) and the Poisson’s ratio \( v_0 \) are given by

\[
v_0 = 1 - \frac{2}{1 + \frac{2}{3}(\alpha + \beta + \gamma) \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right)}.
\]

\[
E_0 = \frac{2\sqrt{3(\alpha + \beta + \gamma)}}{3 \left[ 1 + \frac{2}{3}(\alpha + \beta + \gamma) \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right) \right]}
\]

Equation (7) for \( v_0 \) allows values between 1/3 and 1 to be realized. It is convenient to use three, rather than two, parameters, because of the symmetry of the triangular network. The extra free parameter can be used as a consistency check. The simulations
reported in this paper were done with \((x, \beta, \gamma) = (1, 1, 1), (1, 1, 4)\) and \((1, 6, 7)\), giving \(v_0 = 1/3, 1/2\) and 0.606, respectively.

When holes are introduced, springs with zero force constant are connected between pairs of hole pixels and across the hole boundary, while \(x-, \beta-\) and \(\gamma\)-springs connect the matrix pixels.

Once the spring network has been set up, a uniaxial strain is applied, typically \(\sim 10^{-3}\). The nodes of the networks and the length of the unit supercell perpendicular to the applied strain are simultaneously relaxed using a specialized conjugate gradient algorithm (DEMBO and STEINHAUG, 1983; DAY, 1990). When the network is fully relaxed, the Young's modulus \(E\) can be immediately obtained from the total elastic energy per unit area, and the Poisson's ratio \(\nu\) from the new length of the unit supercell perpendicular to the applied strain. Equal tensile and compressive strains are used to average out any non-linearities, which were found to be small for strains \(\sim 10^{-3}\). Non-linearities are present because the (harmonic) springs in the network are not collinear. Some test results on the algorithm and digitization scheme are described in the Appendix.

BREAKDOWN MODES FOR THE NECKS

We will refer to the matrix material between two adjacent circles as a neck, as shown in Fig. 3(1). When adjacent necks come close, they become soft elastically. This means that most of the strain energy is deposited in the neck regions when the sample is subject to an external load. As we will see, the energy is localized in a

![Diagram](image)

Fig. 3. Three possible breakdown modes of the necks between adjacent circles: (1) compression, (2) shear bend, and (3) bend.
remarkably small area around the center of the neck (HALPERIN et al., 1985) which has width \( w \) and an effective length \( \sqrt{wR} \), where \( R \) is the radius of the circle shown in Fig. 1(a)–(c). The three possible breakdown modes are shown in Fig. 3 under: (1) compression, (2) shear bend, and (3) bend. We refer to these as breakdown modes as they are responsible for the vanishing of the elastic moduli at percolation. We will discuss each of these three modes in turn. By doing appropriate integrations over the neck region, asymptotically exact formulae can be found for each case. The \( x \)-axis is drawn through the center of the neck, with the origin at the narrowest part. The width of the throat \( W \), a distance \( x \) from the center, is given by

\[
W = w + x^2/R
\]

to leading order, i.e. we regard the circle as being parabolic in the throat region.

**Compression**

If the neck region is loaded in compression as shown in Fig. 3(1), we can adapt the standard expression for the uniaxial compression of a beam:

\[
\Delta u = \frac{F \, dx}{E_0 \, W},
\]

where \( \Delta u \) is the longitudinal displacement caused by the force \( F \). The 2d Young's modulus of the beam is \( E_0 \). Using (9), and integrating over the beam, we obtain

\[
u = \frac{F}{E_0} \int_{-\infty}^{+\infty} \frac{dx}{w + x^2/R} = \frac{\pi F}{E_0 \, \sqrt{w}},
\]

where we have extended the integration limits to infinity, as all the strain is concentrated near the origin and no error is introduced by extending the integration range when we are interested in the asymptotic limit. Comparing (11) with (10), we see that the neck acts as if it were a beam of uniform width \( w \) and length \( l_{w} = \pi \sqrt{wR} \). Writing \( F = \alpha_{c} \, u \), we define an effective force constant \( \alpha_{c} \) for neck compression:

\[
\alpha_{c} = \frac{E_0}{\pi} \left( \frac{w}{R} \right)^{1/2}.
\]

**Shear bend**

The shear bend breakdown mechanism is shown in Fig. 3(2). The beam is sheared with forces \( F \) at either end. Torques \( T \) are also applied at each end in order to keep the two ends parallel, which is accomplished if \( T = FL \), where \( 2L \) is the full length of the beam. This length will be allowed to tend to infinity. The bending moment a distance \( x \) from the center of the beam is \( F(L-x) \), so that the displacement \( y \) of the central axis is governed by the second-order differential equation for a beam:

\[
\frac{d^2y}{dx^2} = \frac{12F(L-x)}{E_0 \, W^3}.
\]
This is the standard beam equation (SOUTHWELL, 1935) with $W^2/12$ being the radius of gyration of a slice through the cross section. Using (9), together with the boundary conditions that $y = 0$ at $x = 0$, and $dy/dx = 0$ at $x = L$, and finally letting $L \to \infty$, we find that

$$y(x) = \frac{3F}{2E_0} \left( \frac{R}{\omega} \right)^{3/2} \left[ \tan^{-1} \left( \frac{x}{\sqrt{\omega R}} \right) + \frac{x\sqrt{\omega R}}{\omega R + x^2} \right].$$  

(14)

This is shown in Fig. 3(2) as the dashed line running along the center of the beam. The result is that the two ends of the beam have a net parallel displacement of $(3F\pi/2E_0)(R/\omega)^{3/2}$ from which we can define an effective force constant $\alpha_v$,

$$\alpha_v = \frac{2E_0}{3\pi} \left( \frac{w}{R} \right)^{3/2}. \tag{15}$$

By comparing (15) with a similar deformation on a beam of uniform width $w$, we can assign an effective length $l_{ef} = (3\pi/16)^{1/3} \sqrt{wR} = 0.84\sqrt{wR}$.

**Bend**

The bending mode of the beam is shown in Fig. 3(3). The beam is subject to torques $T$ at each end as indicated. The length is again allowed to tend to infinity. The displacement of the center line $y$ is governed by the beam equation

$$\frac{d^3y}{dx^2} = \frac{12T}{E_0W^3}. \tag{16}$$

Using (9), together with the boundary conditions that $y = dy/dx = 0$ at $x = 0$, we find that

$$y(x) = \frac{3TR}{2E_0\omega^2} \left[ \frac{3x}{\sqrt{\omega R}} \tan^{-1} \left( \frac{x}{\sqrt{\omega R}} \right) + \frac{x^2}{\omega R + x^2} \right]. \tag{17}$$

This is shown in Fig. 3(3) as the dashed line along the center of the beam. The net result is that the beam is bent by an angle $T/\gamma_b$, where the angular force constant $\gamma_b$ for bending is

$$\gamma_b = \frac{2E_0R^2}{9\pi} \left( \frac{w}{R} \right)^{3/2}. \tag{18}$$

This result has previously been obtained by HALPERIN et al. (1985) and we include the full derivation here together with the amplitude. This will be needed when discussing the critical behavior and breakdown mode of the random hole case (c). Comparing the result (18) to that for a uniform beam, we obtain an effective length $l_{ef} = (3\pi/16)^{1/3} \sqrt{wR} = 0.59\sqrt{wR}$.

We note that all the results of this section are asymptotically exact as the necks become very narrow, when they behave in all cases as if they had an effective length $\sim \sqrt{wR}$. All the strain energy is concentrated in this neck region which we find it convenient to envisage acting like a piece of soft chewing gum. Of course, this is not
to suggest that the necks are viscoelastic, as we are working entirely within linearized elasticity theory. We note that the numerical coefficient preceding $\sqrt{wR}$ in the effective length $l_{eq}$ is about a factor 4 smaller for the shear bend and the bend, compared to the compression.

**Critical Behavior**

Percolation occurs for cases (a) and (b) when the circular holes touch. For the honeycomb arrangement of holes (a), this leads to a critical area fraction of matrix material:

$$p_c = 1 - \pi/(3\sqrt{3}) = 0.3954,$$

while, for the triangular arrangement of holes (b),

$$p_c = 1 - \pi/(2\sqrt{3}) = 0.0931.$$  \hspace{1cm} (19)

In both regular cases, close to percolation, the width of the neck $w$ is linear in $p - p_c$:

$$\frac{w}{R} = \frac{p - p_c}{1 - p_c},$$

where $R$ is set equal to $R_c$, the radius of the hole at percolation.

In case (a), holes in a honeycomb arrangement, the necks form a triangular lattice and the system becomes equivalent to a triangular net of central-force springs with spring constant $\alpha_c$ given by (12). Combining (12) and (21), we find that

$$\alpha_c = \frac{E_0}{\pi} \left(\frac{p - p_c}{1 - p_c}\right)^{1/2}.$$  \hspace{1cm} (22)

We know that for the triangular network with nearest-neighbor forces $\alpha_c$, all the elastic moduli are proportional to $\alpha_c$ (FENG et al., 1985). In particular, the Young's modulus $E = 2\alpha_c/\sqrt{3}$ and the Poisson's ratio $\nu = 1/3$. Therefore, for case (a) in the critical region, we find that

$$\frac{E}{E_0} \approx \frac{2}{\pi\sqrt{3}} \left(\frac{p - p_c}{1 - p_c}\right)^{1/2} \quad \text{and} \quad \nu \approx 1/3,$$

where $p_c$ is given by (19).

For case (b), holes in a triangular close packed arrangement, the necks form a honeycomb lattice. The honeycomb lattice has most of its elastic moduli zero with only central forces (BERGMANN, 1985; GIBSON and ASHBY, 1988). An exception is the bulk modulus $K = a_c/\sqrt{12}$, which, using (12) and (21), gives the result

$$\frac{K}{E_0} \approx \frac{1}{2\pi\sqrt{3}} \left(\frac{p - p_c}{1 - p_c}\right)^{1/2}.$$  \hspace{1cm} (24)

The shear and Young's moduli do not involve any compression of the necks, but
rather are controlled by the shear bend mechanism. Combining (15) and (21), we have

\[ x_{ab} = \frac{2E_0}{3\pi}\left(\frac{p-p_e}{1-p_e}\right)^{3/2}. \]  

(25)

The Young’s modulus \( E = 4\pi x_{ab}/\sqrt{3} \), so that, in the critical region for case (b),

\[ \frac{E}{E_0} \approx \frac{8}{3\pi\sqrt{3}}\left(\frac{p-p_e}{1-p_e}\right)^{3/2}. \]  

(26)

and the Poisson’s ratio \( v = 1 - E/2K \) can be obtained from (24) and (26):

\[ v \approx 1 - \frac{16}{\pi\sqrt{3}}(p-p_e). \]  

(27)

The critical behavior of the triangular array of holes is richer than the honeycomb grid, because two distinct breakdown modes are involved. This is the reason for the flow to a critical Poisson’s ratio of 1 rather than 1/3. We will discuss these results further in the comparison with the simulation data. Note that the Poisson’s ratio as used in this paper is the area Poisson’s ratio, which is bounded from above by 1, rather than by 1/2 as is familiar in three-dimensional systems.

For the random case (c), the critical exponent can be predicted to be 5.46 (Halperin et al., 1985). This value is incremented from 3.96 (Zabolitzky et al., 1986) when the necks have a minimum fixed width, by an amount equal to 5/2−1, allowing for the fact that the necks can be infinitesimal narrow. The 5/2 arises from the bending mode of the beam discussed earlier.

**THE DILUTE LIMIT AND EMT**

The problem of a single circular hole can be solved exactly, and this single-inclusion result forms the basis for EMTs. The simplest EMT for circular holes (Thorpe and Sen, 1985) in a sheet leads to the results

\[ E/E_0 = (3p-2), \]  

(28)

\[ v - \frac{1}{3} = (v_0 - \frac{1}{3})(3p-2). \]  

(29)

The other elastic constants can be obtained by inverting (5) and (6). The results (28) and (29) are remarkably simple, and are exact in the dilute limit (see the Appendix) for small \((1-p)\), where it can be seen that \(E/E_0\) is (unexpectedly) independent of \(v_0\). Indeed, (28) predicts that \(E/E_0\) is independent of \(v_0\) for all \(p\). However, (28) and (29) incorrectly predict the percolation threshold \(p_e\) to be 2/3. In reality, percolation occurs at \(p_e \approx 0.32\) (Gawlik and Stanley, 1981; Xia and Thorpe, 1988). At percolation, the critical Poisson’s ratio reaches a universal value of 1/3 (as predicted by an EMT) independent of the Poisson’s ratio for the matrix material \(v_0\). Similar results have been found in lattice systems (Garboczi and Thorpe, 1985; Bergmann, 1985). This
result can be understood by noting that any external loading will produce similar uniaxial strains in the narrow necks between adjacent circles at percolation.

SIMULATION RESULTS

Honeycomb arrangement of holes

In Figs 4 and 5, we show the results for the Young’s modulus $E/E_0$ and the Poisson’s ratio $v$ for the honeycomb arrangement of holes. The exact results in the dilute limit (A1) and (A2) and in the critical region (23) are shown by dashed lines.

We have found it convenient to summarize our results using a fairly complex interpolation formula similar but different to formulae used previously (Tobochnik et al., 1989). These formulae incorporate the exactly known results, i.e. the initial slope, the percolation concentration and the critical exponent. It is useful to introduce the additional parameter $\alpha$ (Garbočzi et al., 1991) because, in cases (a) and (b), we also know the critical amplitude via (23) and (26). We also need a free fitting parameter $\theta$ to get acceptable agreement with the simulation data. The expression we use is

$$
\frac{E}{E_0} = \left\{ \left[ \frac{1-q/q_c}{1+q/(mq_c)} \right] \left[ 1 + q/q_c + \alpha \theta (q/q_c)^2 + \alpha (1-\theta)(q/q_c) \right] \right\}^m, \quad (30)
$$

where $q = 1-p$ etc. Using $q_i = 1-p_i = 1/3$, $q_c = 1-p_c = \pi/(3\sqrt{3})$ and $m = 1/2$ from (23), the parameter $\alpha = -1.375$ to get the correct critical amplitude in (23), and the fitting parameter $\theta = 2.19$ for the best fit to the simulation data. This formula is shown

![Graph showing simulation data for the Young's modulus $E/E_0$ for various values of the Poisson's ratio $\nu_o$ of the matrix material for a regular array of circular holes in a honeycomb arrangement [case (a)]. The dashed lines are the exact results in the dilute and critical regions. The solid line is the fit to the data using the interpolation formula (30).](image)
plotted as the solid line in Fig. 4, and is seen to summarize our results in an accurate and useful way.

**Triangular arrangement of holes**

In Figs 6 and 7, we show the results for the Young’s modulus $E/E_0$ and the Poisson’s ratio $v$ for the triangular close packed arrangement of holes. The exact results in the dilute limit (A1) and (A2) and in the critical region (26) and (27) are shown by dashed lines.

We have again used the interpolation formula (30) to fit our data, putting $q_t = 1 - p_t = 1/3$, $q_c = 1 - p_c = \pi/(2\sqrt{3})$ and $m = 3/2$ from (26); the parameter $\alpha = -0.2510$ to get the correct critical amplitude in (26); and the free parameter $\theta = -1.83$ for the best fit to the simulation data. This formula is shown plotted as the solid line on Fig. 6 and is also seen to summarize our results in an accurate and useful way.

**Random arrangement of holes**

Figures 8 and 9 present the results for $E/E_0$ and $v$ plotted against $p$, averaged over 10 configurations, and computed for the three choices of the force constants (and hence $v_0$) given above. The points are the simulation results, and the dashed lines are the effective-medium results (28) and (29). The solid line in Fig. 8 is an interpolation formula fit to the simulation data points, and is discussed in the next section. In Fig. 8 it is interesting to note that it appears that $E/E_0$ is independent of $v_0$ for all values
Fig. 6. Simulation data for the Young’s modulus $E/E_0$ for various values of the Poisson’s ratio $\nu_0$ of the matrix material for a regular array of circular holes in a triangular arrangement [case (b)]. The dashed lines are the exact results in the dilute and critical regions. The solid line is the fit to the data using the interpolation formula (30).

Fig. 7. Poisson’s ratio $\nu$ plotted against $p$ for various values of the Poisson’s ratio $\nu_0$ of the matrix material for a regular array of circular holes in a triangular arrangement [case (b)]. The dashed lines are the exact results in the dilute and critical regions. The fixed point is marked by a star.
Fig. 8. Simulation data for the Young's modulus $E/E_0$ for various values of the Poisson's ratio $\nu_0$ of the matrix material for a random arrangement of holes [case (c)]. The straight line is the EMT result (28). The solid line is the interpolation formula (32), with $m = 3.4$.

Fig. 9. Poisson's ratio $\nu$ plotted against $p$ for various values of the Poisson's ratio $\nu_0$ of the matrix material for a random arrangement of holes [case (c)]. The dashed lines are the EMT results (29).

of $p$, and not just for $p$ close to 1. We have independently computed $K/K_0$ and $\mu/\mu_0$, and have seen that they are clearly not independent of $\nu_0$. Figure 9 shows the $\nu$ flows to a fixed point of about 1/3, as $p$ decreases. This is consistent with the 1/3 predicted by an EMT in (29), but not conclusive. Interestingly, the flow to the fixed point is essentially complete by $p = 2/3$, the EMT percolation threshold, which is well above $p = 0.32$, the true percolation threshold (Gawlinski and Stanley, 1981; Xia and Thorpe, 1988). The error bars on the Poisson's ratio are large for $p < 0.6$. These are the statistical error bars from the average over 10 configurations. The error bars of the Young's modulus are much smaller, as can be seen from Fig. 8. Although the Poisson's ratio is computed directly, it can be regarded as the ratio of two elastic moduli, both of which become very small for $p < 0.6$. A preliminary account of our results for the random case (c) has been previously published by us (Day et al., 1991).

For $D = 11$ pixel circles in a 210 by 210 lattice, we have directly computed the
percolation threshold, using a lattice "burning algorithm" (Stauffer, 1985), and found it to be $p_c = 0.34 \pm 0.03$, in good agreement with the known result that was obtained using a true continuum simulation (Gawlinski and Stanley, 1981).

The area fraction of matrix material remaining, $p$, is related to the number of holes per unit area, $n$, via (2). This is a statistical relationship that holds if a sufficiently large number of holes are present (Xia and Thorpe, 1988). We have checked (2) in analyzing our data, since $p$ may be evaluated directly by counting the number of matrix pixels remaining, and found (2) to hold.

In Fig. 10, we have collected together our results for the Young's modulus for cases (a)–(c). As expected the different geometries have a profound effect upon the behavior of the elastic modulus outside the dilute regime of small $(1-p)$. Indeed the subsequent behavior is largely controlled by the location of the critical point and the critical exponent. All three curves in Fig. 10 start out with the same initial slope (corresponding to $p_c = 2/3$) in the dilute limit; however, all have different critical exponents as well as different critical points. It is interesting to note that the results for the regular cases (a) and (b) run together up to $p \approx 0.6$. By expanding (30) up to quadratic terms in $q^2$,

$$\frac{E}{E_0} = 1 - 3q + cq^2 + O(q^3),$$

where, using values for $\alpha$ and $\theta$ from fitting our simulation data, we find that $c = 8.0$ for case (a) and $c = 6.5$ for case (b), which are fairly close to one another. Because of our fitting procedure, these numbers can only be taken as rough estimations. By plotting $E/E_0 - 1 + 3q$ against $q^2$, we estimate that $c = 4.2 \pm 0.2$ for both case (a) and

Fig. 10. Comparison of simulation results for the Young's modulus $E/E_0$ for: (a) the regular honeycomb network, (b) the regular triangular network, and (c) the random arrangement. The dashed line is the exact dilute result.
case (b) in the range \( q < 0.25 \). The uncertainty in the value of the coefficient \( c \) in (31) is associated with the range of \( q \) over which a fit is required.

The location of the critical point is of little help in determining the ordering of the curves as the critical point for the random case (c) lies between the critical points for the regular cases (a) and (b), but the Young’s modulus for (c) lies below the other two for most of the range.

We note that the asymptotic forms for the Poisson’s ratios in Figs 5 and 7 are not attained quite as well as for the Young’s moduli in Figs 4 and 6. We have checked that the asymptotic forms for the bulk moduli, which have a critical exponent of 1/2 for cases (a) and (b), are in good agreement with the simulations. Our conclusion is that the asymptotic form for the Poisson’s ratio is valid over a smaller region than for the moduli themselves. The failure of the Poisson’s ratio to converge to exactly 1/3 in case (a) as expected is due to the finite size of the mesh used in the simulation.

**Comparison with Experiment**

Recent experiments (Sofo et al., 1987; Lobb and Forrester, 1987) have measured the elastic moduli of square sheets, containing randomly centered circular holes, under uniaxial loading. Because of the clamping arrangement at the ends, a modulus intermediate between Young’s modulus \( E \) (free sides) and \( C_{11} = E(1-\nu^2) \) (clamped sides) is obtained. For simplicity, we compare our results for \( E/E_0 \) with the experiments in Fig. 11. This should be a good approximation, as our numerical results show that, if \( \nu_0 = 1/3 \), then \( \nu \approx 1/3 \) for all \( p \). The materials used in the experiments (Sofo et al., 1987; Lobb and Forrester, 1987) would be expected to have an area Poisson’s ratio close to 1/3. To facilitate comparison between our simulations and the experiments, we have fitted our results to the empirical form

\[
\frac{E}{E_0} = \left[ 1 - \frac{q}{m \rho_l} - \frac{q^2 (mq_l^2 - q_0^2)}{m q_0^2 \rho_c} \right]^w,
\]

(32)

Fig. 11. Comparison of the simulation results for the Young’s modulus \( E/E_0 \), as represented by the interpolation formula (32), shown as a solid line, with the experiments of Lobb and Forrester (1987) (circles) and Sofo et al. (1987) (squares). These results are for case (c), a random arrangement of holes.
where we have set the initial slope parameter \( q_t = 1 - p_t = 1/3 \), the percolation threshold \( q_c = 1 - p_c = 0.66 \), to match our numerical results; and the exponent \( m = 3.4 \) is obtained by a least-squares fit to the simulation data. The experimental data points have been digitized from the published graphs. Form (32) is equivalent to formula A used previously (Xia and Thorpe, 1988) and is simpler than (30). An extra parameter was used in (30), because we know the critical amplitudes for the regular cases.

The exponent \( m \) is not meant to be a critical exponent, as our data are quite far from the critical region near \( p_c \). Rather, \( m \) is merely a free parameter in the interpolation formula (32). It is very difficult to obtain good results in the critical region as \( E/E_0 \) is so small. On theoretical grounds, the true critical exponent \( f \) is expected to be about 5.46 (Zabolitzky et al., 1986; Halperin et al., 1985; Sofo et al., 1987; Lobb and Forrester, 1987) although to date the experimental and now simulation data supporting this value is underwhelming. Such large critical exponents are exceedingly hard to verify in either real experiments or in simulations.

The interpolation formula (32) fits the data in Fig. 8 well. Figure 11 shows that the interpolation formula, with the same exponent \( m \), also fits the data from Sofo et al. (1987). For reasons that are not clear to us, the data from Lobb and Forrester (1987) are significantly larger than our results for the laser values of \( \mu \).

**Summary**

We have shown that this simplest of finite-element algorithms, a discretized-spring scheme based on linear Hooke springs, can give good results in 2D composites. Such calculations are now feasible with the use of modern supercomputers. The calculations reported in this paper took \( \sim 170 \) h of CPU time on a CYBER 205 supercomputer, with most of this time, \( \sim 150 \) h, being devoted to the random case (c). It is important to emphasize that the present simple scheme, combined with averages over many samples to give good statistics, is best suited to the calculation of the elastic behavior of random composites. More sophisticated grids and algorithms have the capability of doing better on any particular realization, but that is not necessary here.

All our results, i.e. simulation data, and exact results in the dilute and critical region, are consistent with the following proposition—the relative Young's modulus \( E/E_0 \) of a 2D sheet containing circular holes, overlapping or not, is the same for all materials with a prescribed geometry. This proposition is unproven at the present time, and it is unknown whether it can be extended to other shapes like ellipses, rectangles, etc. This proposition is not true for rigid or other inclusions in two dimensions (Thorpe and Sen, 1985; Jasiuk et al., 1992) or for any kinds of inclusions at all in 3D (Christensen, 1979).

In the past, various EMTs have been compared with one another and checked against various exact bounds on the elastic constants (Watt et al., 1976). This is rather unsatisfactory. The present work opens up the possibility of producing high-quality numerical results on well-controlled composite geometries, in order to evaluate which, if any, EMTs are valid away from the dilute limit. A direct comparison between EMTs and real experiments is always dangerous because of the uncertainties present.
in the characterization of the geometry and other parameters of real composites. Thus, computer simulation can form an important bridge between approximate EMTs and experiments.

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APPENDIX

We have performed some tests on the random hole case (c) in order to establish the algorithm. In doing this, we have studied the dependence of the Young's modulus $E$ and the Poisson's ratio $\nu$ on $p$, the area fraction of the matrix material remaining, for three different choices of $\alpha$, $\beta$ and $\gamma$. These are $(\alpha, \beta, \gamma) = (1, 1, 1), (1, 1, 4)$ and $(1, 6, 7)$ giving Poisson's ratios of $1/3, 1/2$ and 0.606.

Two checks can be made on the accuracy of this digital-image approach. The first check is on whether we are really simulating a continuum system or not. The results for given values of $E_0$ and $\nu_0$ should not depend on the specific choice of lattice force constants, if we are truly carrying out a continuum simulation. Table A1 shows the results for $E/E_0$ and $\nu$ at $p = 0.7377$, averaged over the same 10 geometrical configurations. The first row is for $\alpha = \beta = 1$ and $\gamma = 4$, the second row interchanges $\alpha$ and $\gamma$, and the third row is for a completely different choice of force constants but giving the same values of $E_0$ and $\nu_0$. The results on these three different systems all agree, to within less than 1%.

The second check is on how accurately the single-hole exact result is reproduced. Equations (28) and (29) give the exact initial slope for the one-defect problem when $(1-p)$ is small:

\[
\delta E = \frac{E}{E_0} - 1 = -3(1-p).
\]

From (A2), we may define $p_l = 2/3$ to be the intercept of the extrapolated initial slope of the Young's modulus with the abscissa.

Table A2 shows the percent difference from the exact result for $\delta E$ and $\delta \sigma$, for three different hole diameters $D = 2R$, and $(\alpha, \beta, \gamma) = (1, 1, 4)$. As expected, the results for $E/E_0$ become better as more pixels are used to define the circular hole. The diameter of 11 pixels used in this

| Table A1. Results for the random case (c) as a check on the algorithm for the elastic moduli, at $p = 0.7377$ |
|---|---|---|---|---|
| $\alpha$ | $\beta$ | $\gamma$ | $E/E_0$ | $\nu$ |
| 1 | 1 | 4 | 0.2899 | 0.3880 |
| 4 | 1 | 1 | 0.2898 | 0.3886 |
| 2 | 3.3 | 0.69 | 0.2916 | 0.3879 |
TABLE A2. Single-hole results as a check on the algorithm in the dilute limit

<table>
<thead>
<tr>
<th>$1-p$</th>
<th>$D = 2R$</th>
<th>$\Delta(\delta E)$ (%)</th>
<th>$\Delta(\delta v)$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00247</td>
<td>11</td>
<td>10.2</td>
<td>0.02</td>
</tr>
<tr>
<td>0.01077</td>
<td>23</td>
<td>3.2</td>
<td>0.03</td>
</tr>
<tr>
<td>0.02220</td>
<td>33</td>
<td>0.2</td>
<td>0.01</td>
</tr>
</tbody>
</table>

paper results in a 10% error in the change in $E$ caused by the introduction of one circular hole. The percent error in $E$ itself is, of course, very much smaller and less than a tenth of a percent. It is interesting to compare this with the equivalent electrical problem. We have recently calculated the effect of one elliptical hole on the conductivity of a conducting sheet, using a square lattice digital-image scheme, and found that the error in the initial slope decreased linearly with the number of pixels used per unit length to represent the hole (Garboczi et al., 1991). Table A2 implies that this error decreases more quickly for the elastic problem, as doubling the number of pixels per unit length caused the error to decrease by more than a factor of 3, and tripling the number of pixels per unit length caused the error to decrease by more than a factor of 10. The surprising result in Table A2 is the fact that, at least for diameters greater than or equal to 11 pixels, the error in the initial slope for the Poisson’s ratio seems to be independent of the number of pixels per unit length, and is essentially a computer rounding-off error.

**Note Added in Proof**

Since this paper was written, we have had extensive correspondence with G. W. Milton, who has supplied us with an unpublished proof of the proposition, given in the abstract and discussed extensively in this paper, that the relative Young’s modulus $E/E_0$ of a 2d sheet containing circular holes, overlapping or not, is the same for all materials, independent of the Poisson’s ratio $\nu_0$, for any prescribed geometry. The Milton proof is based on the work of Lurie and Cherkaev [1986, Usp. Mek. (Adv. Mech.) 9, 3 (in Russian)], and we state the result below.

Suppose that a 2d composite material has continuously varying bulk and shear moduli given by $K(r)$ and $\mu(r)$, and that the effective moduli of the material are $K^*$ and $\mu^*$, then if

$$\frac{1}{K'(r)} = \frac{1}{K(r)} - 2\lambda$$ and $$\frac{1}{\mu'(r)} = \frac{1}{\mu(r)} + 2\lambda$$

where $\lambda$ is a constant, then

$$\frac{1}{K^*} = \frac{1}{K} - 2\lambda$$ and $$\frac{1}{\mu^*} = \frac{1}{\mu} + 2\lambda$$

Because of the form of the Young’s modulus in two dimensions as given in (5), the above transformation can be recast to give an invariant Young’s modulus. In the case of holes (of any shape, not just circular), we thus are led to the proposed result.

Specialising the result to the hole case, where the holes are characterised by the parameter $p$ as used in the text, we may write the result as

$$E'(p)/E_0 = E(p)/E_0$$

and

$$\nu'(p) - \nu(p) = (\nu_0 - \nu_0)(E(p)/E_0),$$

where we have used the values of the Poisson’s ratio $\nu_0 = \nu(1)$ and Young’s modulus $E_0 = E(1)$ at $p = 1$ to eliminate the (unknown) constant $\lambda$. This provides the first rigorous proof of the conjectured flow (see, for example, Schwartz et al., 1985, Phys. Rev. B32, 4607) of the Poisson’s ratio to a fixed point at a percolation threshold, defined as that value of $p$ at which $E(p)$ first goes to zero. We have used the interpolation expressions in the text for the Young’s
modulus, (30) and (32), and also a fitted value for a single value of the initial Poisson's ratio, to obtain an expression for \( v(p) \) for all theoretically allowed values of the Poisson's ratio in the range \(-1 < v_0 < 1\). For the triangular and random lattices, we have chosen \( v_0 = 1/3 \) for which from our numerical results we have \( v(p) = 1/3 \) for all \( p_c < p < 1 \) as shown in Figs 5 and 9. For the honeycomb lattice, we fit the \( v_0 = 1/3 \) curve with the interpolation formula

\[
v(p) = \frac{1}{3} \left( 1 - \left[ \frac{(1-p)}{(1-p_c)} \right]^n \right) + \left[ 1 + \alpha (p-p_c) \right] 
\]

where \( n = 4 \), and \( \alpha = 0.9676 \). All three flow diagrams are shown in Fig. 12.

Chen and Thorpe (unpublished) have recently used this numerical approach to obtain results for non-overlapping circular holes. The results, when plotted against \( p \), the matrix area fraction, are quite similar to the overlapping circles shown in Figs 9 and 10, except that the Young's modulus is a little smaller (by \( \sim 5\% \) at \( p = 0.7 \)).

Fig. 12. Flow diagrams for the Poisson's ratio for: (a) the regular triangular network, (b) the regular honeycomb network, and (c) the random arrangement, using the argument of Milton as given in the Note Added in Proof. The dashed line with a Poisson's ratio \( v_0 = 1/3 \) in each graph was used as the reference as described in the text.