Comment on "Percolation in Isotropic Elastic Media"

In a recent Letter, Garcia-Molina, Guinea, and Louis argued that the percolation threshold and the critical properties of a model for rigidity percolation in an isotropic elastic medium depend strongly on the force constants. We give a general proof that shows this result to be incorrect and in contradiction of many known results on similar systems.

Consider an elastic network of Hook springs with the potential

\[ V = \frac{1}{2} \sum_{i,j} K_{ij} [(u_i - u_j) \cdot \hat{r}_{ij}]^2, \]

which describes a network of \( N \) point masses connected by springs \( K_{ij} > 0 \) where the \( u_i \) are the site displacements from their equilibrium positions. Many examples of such networks have been studied where the springs \( K_{ij} \) are randomly present in a lattice model. For any given set of \( K_{ij} \) the system will have a set of normal modes, \( N_0 \) of which will be zero-frequency modes. Suppose that we identify the displacements associated with a single zero-frequency mode as \( u^0_i \). Then, as this deformation costs no energy and because the potential (1) is a sum of positive terms, we must have

\[ (u_i^0 - u_j^0) \cdot \hat{r}_{ij} = 0. \]

Hence if any of the \( K_{ij} > 0 \) is increased or decreased, this zero-frequency mode will remain a zero-frequency mode. Extending this argument we see that the number of zero-frequency modes cannot change as the magnitudes of the \( K_{ij} \) are varied and that the vector space of all possible deformations is divided into two invariant subspaces, \( S \) and \( S' \), where \( S \) has dimension \( N_0 \) and is spanned by the eigenvectors of the zero-frequency modes and \( S' \) has dimension \( dN - N_0 \) and is spanned by the eigenvectors of those modes with finite frequencies.

If we apply a stress \( F \) at the surface and calculate the resultant displacement \( U \), where \( F \) and \( U \) are \( dN \)-dimensional vectors, then the elastic modulus \( C \) associated with \( F \) is defined as

\[ C = \frac{1}{2} \frac{F \cdot U}{U \cdot U}. \]

If \( F \) couples to any of the zero-frequency modes, i.e., \( F \) has components in the subspace \( S \), then the elastic modulus is zero. If \( F \) only has components in the subspace \( S' \), the displacement \( U \) costs energy and the elastic modulus is positive. As the subspaces do not change as the \( K_{ij} > 0 \) are varied, so the elastic modulus corresponding to a fixed applied \( F \) cannot change from zero to positive or vice versa. An immediate result of these considerations is that the percolation threshold \( \rho_{\text{cen}} \) cannot be shifted by varying the \( K_{ij} \).

Garboczi and Thorpe have studied a square net with nearest-neighbor springs \( a \) and next-nearest-neighbor springs \( \gamma \), where both are present with probability \( p \). They find that \( \rho_{\text{cen}} = 0.5 \) as long as both \( a, \gamma > 0 \) and there is a crossover to a discontinuous change in the bulk modulus at \( p = 1 \) as either \( a \) or \( \gamma \) becomes zero. The example given in Ref. 1 should exhibit a similar crossover from the rigidity threshold of the triangular net at \( \rho_{\text{cen}} = 0.65 \) when the two spring constants \( k_a, k_b > 0 \) to a discontinuous transition of the bulk modulus of the honeycomb lattice at \( p = 1 \) when \( k_b = 0 \). We expect the critical exponents to exhibit the usual crossover behavior and not change continuously as claimed by the authors of Ref. 1; indeed, the variation of the ratio of critical exponents \( f/v_x \) is almost certainly an artifact of their incorrect identification of the elastic rigidity threshold.

Although the finite-size scaling technique used in Ref. 1 can be very powerful, it is necessary to plot the intersection of more than two curves and monitor any systematic size dependence which we would expect to be particularly important when studying the crossover from a continuous to a discontinuous transition. It is our experience that the system is not fully relaxed until the maximum force on any site is several orders of magnitude smaller than the cutoff criterion used in Ref. 1.

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