Percolation of elastic networks under tension

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We examine the elastic properties of random networks of Hooke springs under a tension supplied by a frame. As an illustration of the general ideas we use the triangular net with a fraction \( p \) of the nearest-neighbor bonds present. When there is no tension, the transition takes place at the threshold for rigidity percolation \( p_{\text{cen}} = \frac{1}{3} \). When the tension is very large (or equivalently the natural length of the springs is zero) the transition takes place at the familiar connectivity percolation \( p_c \approx \frac{1}{4} \). As the tension is varied, the phase boundary evolves smoothly and continuously between these two limits. We give a general discussion of the elasticity of random networks under tension and present extensive computer simulations for all the relevant quantities on the bond depleted triangular network. The asymmetric (nonrotationally invariant) terms in the elastic moduli tensor are equal to the tension. In general, the Cauchy relations are not obeyed in these networks. Although these networks are nonlinear, we show that a harmonic approximation can be made that leads to a very good effective-medium theory of the phase boundary.

I. INTRODUCTION

It is widely accepted that conductivity percolation and rigidity percolation belong to two different universality classes.\(^1,2\) The percolation threshold \( p_{\text{cen}} \) of a central force network, at which the elasticity vanishes, is much higher than \( p_c \), at which a resistor network loses its conductivity.\(^3\) These two systems also have quite different critical exponents in the critical region, near \( p_{\text{cen}} \) and \( p_c \), respectively. Numerical simulation results\(^4,5,6\) strongly support this classification.

In a recent paper,\(^7\) henceforth called I, we have shown that if a network of Hooke springs with natural length zero are stretched onto a frame, then there is an exact mapping onto the random resistor network problem. This leads to the most important result that the transition takes place at the familiar connectivity percolation \( p_c \). There is also a mapping between all the other relevant quantities. In this limit, the spring problem becomes identical to the conductivity problem. This can be understood by noticing that percolation, in either the resistor network or the central force network with natural length zero, is determined by the geometrical connections. As long as there is a percolating cluster, the current always can flow from one side to the other. Similarly, there is always energy stored in the mapped elastic network through the backbone. This feature is different from the unstrained central force networks studied by many other authors.\(^1,2,4-6\) The networks they used are such that every spring has its natural length in the absence of external stress. The reader is referred to I for more details. There is also some discussion of this important limiting case in this paper. In this article we are concerned with how these results evolve from one to the other as the tension is varied. It is interesting that such a bridge exists between these two classes of problems.

Based on the above considerations, a new model—the "stretched-spring" model—is formed naturally, which can continuously span the gap between conductivity percolation and rigidity percolation. This is quite different from adding angular forces to the central force model which leads to a crossover effect.\(^1\) Our model is defined as follows. On a lattice background with spacing \( L \), we place between nearest neighbors a Hooke's law spring with natural length \( L_0 \), where in general \( L > L_0 \). The area of the lattice is maintained by a frame or by imposing periodic boundary conditions. The former is easier to visualize but the latter are more convenient in practice and are used in our numerical simulations. In the equilibrium state, no net force is exerted on any site, but the springs around a site will be stretched. This can be considered as an internal strain. The physical behavior of such systems is very interesting and controversial\(^8\) and may be relevant to real systems. For instance, the percolation threshold \( p^* \) as a function of the tension is perhaps the most interesting quantity.

We already know through constraint counting\(^9\) and effective-medium theory as well as from numerical simulations\(^5\) that there are two special cases: \( L_0 = 0 \), which mapped to conductivity percolation as shown in I, and \( L_0 = L \), which is the pure central force network.\(^1\) The effective-medium theory\(^4\) gives the thresholds corresponding to these two cases:

\[
p^*(L_0 = 0) = p_c = \frac{2}{z}
\]

and

\[
p^*(L_0 = L) = p_{\text{cen}} = \frac{2d}{z},
\]
where $z$ is the coordination number and $d$ is the dimensionality. These are very close to the exact result $p_c = 2 \sin(\pi/18) = 0.3473$ and the simulation result $p_{\text{cen}} \approx 0.65$. The question is how $p^*$ evolves between these two limits when $L_0 < L$ takes an arbitrary value.

Feng and Sen observed a crossover in their numerical simulations using a dilute Born model. They found that when the contribution of the central force part is dominant, there is a strong crossover from isotropic-force-like behavior near $p = p_c$ to central-force-like behavior near $p = p_{\text{cen}}$. As we will see in Sec. II, this corresponds to the case, in our stretched-spring model, when the natural length $L_0$ increases to the lattice space $L$. However, it should be pointed out the dilute Born model and and the stretched-spring model are different at least in two respects. First, the Born model is actually the contribution to elastic energy in the second order of the displacement, where the isotropic term is not rotationally invariant. In the stretched-spring model, we are considering the complete potential in the system which is rotationally invariant since only the distances $r_{ij}$ between sites are involved. Of course, we have to rotate the frame together with the lattice in order to achieve this rotational invariance. In practice it is convenient to keep the frame fixed so that the system loses rotational invariance. The effects of the fixed frame can be allowed for equivalently the theory and simulations are done by imposing periodic boundary conditions which provide the tension. Secondly, all the displacements in the Born model are relative to the lattice points, while in the stretched-spring model, the displacements are defined relative to the new equilibrium positions. There is also an important conceptual advantage in the stretched-spring model because the tension enters in a natural way into the determination of internal stresses and therefore into the determination of elastic constants.

This paper is organized as follows. Section II presents the elasticity analysis for a perfect triangular net under tension. Section III describes the diluted network under tension and gives details and results of the simulations. Section IV describes some relations arising from symmetry that reduce the number of independent second-order elastic constants. Section V presents an effective-medium theory for the stretched network. This is rather more involved than previous cases because even though the individual springs are linear, the whole system is not.

### II. ELASTICITY OF PERFECT TRIANGULAR NET UNDER TENSION

We start from an elastic network on a pure lattice where all bonds are Hooke's-law springs with force constant $K$. The natural length of each spring is $L_0$ and the available lattice space is $L$. Except for the case $L_0 = L$, the network is stretched ($0 \leq L_0/L \leq 1$). The elastic potential for this model is

$$V = \frac{1}{2} \sum_{(i,j)} K_{ij} (L_{ij} - L_0)^2,$$

where $L_{ij}$ is the distance between the sites at $i$ and $j$ which is $L$ before dilution. We introduce the vector $u_{ij}$ for the small displacement of site $i$ relative to its equilibrium position. We are interested in the case where all the $K_{ij} = K$ for nearest neighbors but will retain the subscripts where it helps to clarify the meaning of an equation. It is noticed that in the perfect lattice case, the equilibrium positions are exactly the corresponding lattice points, while in the diluted case, these two can be far away from each other. The potential can be expanded about the equilibrium position in terms of $u_{ij} = u_i - u_j$. To the second order of $u_{ij}$, we get

$$V = \frac{1}{2} K \sum_{(i,j)} (L_{ij} - L_0)^2 + K \sum_{(i,j)} (L_{ij} - L_0) (u_{ij} \cdot \hat{r}_{ij})$$

$$+ \frac{1}{2} K \sum_{(i,j)} \left[ \left( 1 - \frac{L_0}{L} \right) u_i^2 + \frac{L_0}{L} (u_{ij} \cdot \hat{r}_{ij})^2 \right],$$

where $\hat{r}_{ij} = \mathbf{R}_{ij}/R_{ij}$ is the unit vector between sites $i$ and $j$.

The first sum in (2) is the static energy of the stretched springs and can be written as

$$V_0 = \frac{1}{4} K N z (L_{ij} - L_0)^2,$$

where $N$ is the total number of sites in the system and $z$ is the coordination number. The second sum in (2) is the linear-order contribution, which is written as

$$V_1 = K (L_{ij} - L_0) \sum_{(i,j)} (u_{ij} \cdot \hat{r}_{ij}).$$

We note that the equilibrium condition for the full potential is

$$\frac{\partial V}{\partial u_i} = 0$$

and for the expansion of the potential in terms of displacement, we have

$$\frac{\partial V}{\partial u_i} = 0.$$

Both of them lead to the same equation

$$\sum_{j} K_{ij} (R_{ij} - L_0 \hat{r}_{ij}) = 0$$

for all $i$.

The last sum in (2), i.e., the quadratic term in $u_{ij}$, is

$$V_2 = \frac{1}{2} K \left( 1 - \frac{L_0}{L} \right) \sum_{(i,j)} u_i^2 + \frac{1}{2} K \frac{L_0}{L} \sum_{(i,j)} (u_{ij} \cdot \hat{r}_{ij})^2.$$

Comparing $V_0$ with the Born model potential

$$V_{\text{Born}} = \frac{1}{2} (\alpha - \beta) \sum_{(i,j)} [(u_{ij} - u_i) \cdot \hat{r}_{ij}]^2 + \frac{1}{2} \beta \sum_{(i,j)} (u_i - u_j)^2,$$

we see the correspondence between the Born model parameters $\alpha, \beta$ and the stretched-spring parameters $L_0/L$ and $K$. 

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\textsuperscript{10} p_c = 2 \sin(\pi/18) = 0.3473 \text{ and the simulation result } p_{\text{cen}} \approx 0.65.

\textsuperscript{11} In the stretched-spring model, we are considering the complete potential in the system which is rotationally invariant since only the distances $r_{ij}$ between sites are involved. Of course, we have to rotate the frame together with the lattice in order to achieve this rotational invariance. In practice it is convenient to keep the frame fixed so that the system loses rotational invariance. The effects of the fixed frame can be allowed for equivalently the theory and simulations are done by imposing periodic boundary conditions which provide the tension. Secondly, all the displacements in the Born model are relative to the lattice points, while in the stretched-spring model, the displacements are defined relative to the new equilibrium positions. There is also an important conceptual advantage in the stretched-spring model because the tension enters in a natural way into the determination of internal stresses and therefore into the determination of elastic constants.
\[ \alpha = K , \quad (3a) \]
\[ \beta = K \left( 1 - \frac{L_0}{L} \right) . \quad (3b) \]

Generally, the strain energy is defined as\textsuperscript{12}
\[ U = \sum_{ab} S_{ab} \varepsilon_{ab} + \frac{1}{2} \sum_{ab \gamma r} C_{ab \gamma r} \varepsilon_{ab} \varepsilon_{\gamma r} , \quad (4) \]
where \( S_{ab} \) is the stress tensor and \( C_{ab \gamma r} \) are the second-order elastic constants. For the perfect Bravais lattice the strains \( \varepsilon_{ab} \) are given in terms of the \textit{homogeneous} distortion:
\[ u_i = \sum_{ab} \varepsilon_{ab} \langle R_i \rangle \hat{\varepsilon} . \quad (5) \]

For an \textit{isotropic external strain}, we have the following expressions for \( S_{ab} \) and \( C_{ab \gamma r} \):
\[ AS_{ab} = K (L - L_0) \sum_{(ij)} R_{ij} r_{ij}^a r_{ij}^b \quad (6) \]
and
\[ AC_{ab \gamma r} = K \frac{L_0}{L} \sum_{(ij)} R_{ij} r_{ij}^a r_{ij}^b r_{ij}^\gamma \]
\[ + K \left( 1 - \frac{L_0}{L} \right) \sum_{(ij)} R_{ij}^2 r_{ij}^a r_{ij}^b \delta_{\gamma r} , \quad (7) \]
where \( A \) is the area of the sample, \( R_{ij} \) is the nearest-neighbor distance \( L \), and \( \hat{\varepsilon}_{ij} \) are the unit vectors.

For the triangular net, we have
\[ A = (\sqrt{3} L^2 / 2) N \quad (8) \]
and
\[ \hat{\varepsilon}_{ij} = (1,0), \quad (-1,0) \]
\[ (\frac{1}{2}, \frac{\sqrt{3}}{2}), \quad (\frac{1}{2}, -\frac{\sqrt{3}}{2}) , \]
\[ (-\frac{1}{2}, \frac{\sqrt{3}}{2}), \quad (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) \quad (9) \]
from which the moments of \( \hat{\varepsilon}_{ij} \) can be worked out:
\[ \langle r_{ij}^a \rangle = 0 , \]
\[ \langle r_{ij}^a r_{ij}^b \rangle = \frac{1}{2} \delta_{ab} , \quad (10) \]
\[ \langle (r_{ij}^a)^2 (r_{ij}^b)^2 \rangle = \frac{1}{4} + \frac{1}{4} \delta_{ab} \quad . \]

Substituting back to (6) and (7), we obtain the stress tensor and the elastic constants. The results are listed in the following where we put \( \eta = L_0 / L \).

\textbf{Stress tensor:}
\[ T = S_{xx} = S_{yy} = \sqrt{3} K (1 - \eta) , \quad (11) \]
\[ S_{xy} = S_{yx} = 0 , \quad (12) \]
where \( S_{xx} \) and \( S_{yy} \) are the external tension \( T \).

\textbf{Second-order elastic constants:}
\[ C_{xxxx} = C_{11} = \frac{\sqrt{3} K}{4} (4 - \eta) , \quad (13) \]
\[ C_{xyxy} = C_{12} = \frac{\sqrt{3} K}{4} - \eta \quad , \quad (14) \]
\[ C_{xxxy} = C_{44} = \frac{\sqrt{3} K}{4} (4 - 3 \eta) , \quad (15) \]
\[ C_{xyyx} = \bar{C}_{44} = \frac{\sqrt{3} K}{4} \eta . \quad (16) \]

These results are shown in Fig. 1. As expected, these elastic constants include the known results for the pure scalar case as given in 1 \((L_0 = 0)\) and the pure central force case \((L_0 = L)\). For the \textit{scalar case} \((\eta = 0)\):
\[ C_{11} = C_{44} = \sqrt{3} K \]
and for the \textit{central force case} \((\eta = 1)\):
\[ C_{11} = 3 C_{44} = 3 \sqrt{3} K / 4 . \]

Since neither \( C_{44} \) nor \( C_{12} \) can be obtained in isolation, we design four other cases of initial strains and combine these results to extract \( \bar{C}_{44} \) and \( C_{12} \). The elastic moduli corresponding to these four cases are respectively,

\textbf{pure shear},
\[ \mu_s = \frac{1}{2} (C_{11} + \bar{C}_{44}) = \frac{\sqrt{3} K}{4} (2 - \eta) ; \quad (17) \]

\textbf{pure rotation},
\[ \mu_r = \frac{1}{2} (C_{11} - C_{12}) = \frac{\sqrt{3} K}{4} (2 - 2 \eta) ; \quad (18) \]

\textbf{bulk compression},
\[ B = \frac{1}{2} (C_{11} + C_{12}) = \frac{\sqrt{3} K}{2} ; \quad (19) \]

\textbf{compression-expansion},
\[ b = \frac{1}{2} (C_{11} - C_{12}) = \frac{\sqrt{3} K}{4} (2 - \eta) . \quad (20) \]

These results are shown in Fig. 2. The external strains patterns necessary to produce these results are rather obvious and are given in detail by Tang.\textsuperscript{13}

From these results we obtain

\begin{center}
\begin{tabular}{c c c c c}
\hline
\hline
\textbf{C}_{11} & \textbf{C}_{44} & \textbf{C}_{12} & \textbf{\bar{C}}_{44} \\
\hline
\hline
\end{tabular}
\end{center}

\textbf{FIG. 1.} \( C_{11}, C_{44}, C_{12} \), and \( \bar{C}_{44} \) are plotted against \( \eta \) for the pure triangular lattice.
III. DILUTED TRIANGULAR NET UNDER TENSION

A. Simulation technique

Special care is needed in studying the diluted network under tension. For the perfect lattice where no bonds are missing, the lattice node serves as the equilibrium position because the forces exerted on a site through adjacent stretched springs balance each other. However, as soon as bonds are removed, this balance is destroyed and the network deforms around the missing springs and the sites move to new equilibrium positions. It is around these new equilibrium positions, which can be far away from the corresponding original lattice nodes, that the sites in the network are vibrating. The elastic constants and the stress tensor are determined from this new equilibrium configuration.

The key point is that before applying any extra external stress, we have to obtain the relaxed configuration of the diluted network. This relaxation is determined by the internal stresses, produced by the dilution. A certain amount of static energy is stored in this relaxed network. The details of the numerical simulation are described below:

(a) Generate a diluted triangular lattice by removing bonds randomly with probability \((1 - p)\) from a triangular lattice. Note that the resulting network is not relaxed for the case \(L_0 < L\).

(b) Relax the network obtained from (a). The displacement of site \(i\) is proportional to the force exerted on it. Specifically, during the iteration procedure, the position of atom \(i\) is determined by the previous configuration through

\[\mathbf{R}_i^{(n+1)} = \mathbf{R}_i^{(n)} + \alpha \mathbf{F}_i^{(n)},\]

where \(\mathbf{F}_i\) is the force component given by

\[\mathbf{F}_i = \sum_j K_{ij}(\mathbf{R}_{ij} - L_0 \mathbf{r}_{ij}),\]

and \(\alpha\) is a number which is adjusted to help with the convergence of the simulation and \(0 < \alpha < 1\). Note that \(K_{ij} = K\) or 0 depending on whether a given bond is present or absent. In this work we chose \(\alpha\) to be 0.3 to 0.5. The iteration process should stop when the force on every site is zero, so that practically, we choose an appropriate small number which is good enough to give the final precision required. For example, a maximum force \(F_{\text{max}} = 10^{-8}\) was used corresponding to a strain \(\varepsilon = 10^{-2}\) in our simulation. The energy of the relaxed network is calculated and written as \(E(p, L_0 / L)\).

(c) Depending on whether \(C_{11}\) or \(C_{44}\) or another elastic moduli is being computed, the coordinates of the sites are initially transformed with the appropriate uniform strain.

(d) The network is then further relaxed as the strain will be inhomogeneous in the depleted network. To separate the linear and second-order contribution, we reverse the sign of external strain \(\varepsilon\) and repeat the procedures (c) and (d). The energy of the relaxed configuration is

\[E^+ = E(p, L_0 / L, \varepsilon > 0) = E_0 + S \varepsilon + \frac{1}{2} C \varepsilon^2,
\]

\[E^- = E(p, L_0 / L, \varepsilon < 0) = E_0 - S \varepsilon + \frac{1}{2} C \varepsilon^2,
\]

where \(S\) and \(C\) stand for linear and second-order coefficients, respectively, given by

\[S = \frac{(E^+ - E_0) - (E^- - E_0)}{2 |\varepsilon|},
\]

\[C = \frac{(E^+ - E_0) + (E^- - E_0)}{\varepsilon^2}.\]

B. Simulation results

In this section we present the results of the computer simulations. All simulations were done on a \(20 \times 22\) site triangular net with periodic boundary conditions. Each network is specified by two parameters: \(L_0 / L\) and \(p\). We investigated the elastic behavior of the network in the \(L_0 / L\) plane in the range \(0 < L_0 / L < 1\) and \(0 < p < 1\). Prior knowledge was available along three lines in this plane. Two of them are along \(L_0 / L = 0\) (see I) and \(L_0 / L = 1\) (see Ref. 4), representing conductivity percolation and rigidity percolation, respectively, and the other is along \(p = 1\) with varying \(L_0 / L\) which was discussed in Sec. II.

First, we present a survey of the results for second-order elastic constants for various \(p\) and \(L_0 / L\). Figure 3 shows the results for \(C_{11}\) and \(C_{44}\) against \(p\) for various \(L_0 / L\). Figure 4 shows the results for \(B, b, \mu_s,\) and \(\mu_r\). All the values at \(p = 1\) agree with those obtained in Sec. II. For example, the bulk moduli for various \(L_0 / L\) are all equal to the same value at \(p = 1\) from Eq. (19). In the intermediate region, i.e., \(p\) less than 1 but much greater
than the threshold $p^*$, these elastic constants decrease roughly linearly (this is especially so for the bulk modulus $B$), which suggests that it may be possible to develop an effective-medium theory for the stretched network. It is important to observe that for the networks with different $L_0/L$, the thresholds $p^*$ at which the elastic constant vanishes are different as revealed by our simulation. This is one of our most important results and was not obvious to us before this work was done. Indeed, in some ways this result is surprising as it is more usual to have a simple crossover from one result to the other as in the central force to isotropic Born model case. The smallest $p^*$ is $p_c$, corresponding to the case of $L_0=0$, which is identical to conductivity percolation as shown in I. With increasing $L_0/L$, we find that $p^*$ increases very slowly at first. After passing through $L_0/L=0.5$, $p^*$ experiences larger and larger changes for the same amount of change in $L_0/L$. Finally $p^*$ reaches $p_{cen}$ when $L_0=L$, as expected.

Upon reflection, this dependence of the threshold $p^*$ on $L_0/L$ is perhaps not so surprising. For example, one can envision a percolated cluster on the lattice background (before relaxation) in which there exists a site $i$ with coordination number two. As sketched in Fig. 5, the two bonds adjacent to this particular site connect the other two parts of the cluster in such a way that they are not aligned in a straight line. If the lattice space $L$ is equal to the natural length of the spring, the current position of $i$ is in equilibrium. Then because the bonds can pivot freely without costing any energy, this structure is not able to transmit any elastic forces. However, if $L_0/L < 0.5$, the equilibrium position would be in the middle between the two parts, with the two springs stretched. Here we just ignore the relaxation of the rest of the system without losing the flavor of the discussion. Similarly, for other values of $L_0/L$, one could imagine many other structures, which would be elastically ineffective on the geometrically percolated cluster for a large value of $L_0/L$, now relaxed to a "good" effective elastic structure for smaller $L_0/L$. This explains why $p^*$ is increasing with $L_0/L$. Because of the random nature of the system, these changes take place continuously rather than in jumps.

Beside these second-order elastic constants, there are two other important quantities needed to describe the stretched spring. They are the zeroth-order constant which is the static energy $E$ and the first-order coefficient which is the external tension $T$. The simulation results for $E$ and $T$ are plotted in Fig. 6. These results are obtained by averaging over ten different networks. For the convenience of investigating the transition thresholds, both $E$ and $T$ are scaled such that their values are 1 for
all $L_0/L$ at $p = 1$. It is clear that the threshold is different for networks with different values of $L_0/L$.

From our computer simulations, we have comprehensively determined the set of six quantities $E$, $T$, $B$, $b$, $\mu_s$, and $\mu_r$ that describe the diluted stretched networks. In the next section we find the minimal set of independent quantities by exploring relations between these six quantities.

FIG. 4. $B$, $\mu_s$, $\mu_r$, and $b$ for diluted triangular network for various values of $\eta$.

FIG. 5. Showing a site $i$ that is twofold coordinated to rigid regions by the solid bonds. The maximum extension possible without straining the bridge occurs along the dashed line.

FIG. 6. Energy $E$ (solid curves) and tension $T$ (dashed curves) for the diluted triangular net, scaled by the pure lattice values where the energy per unit area is $\sqrt{3}K(1-\eta)^2$ and the tension is $\sqrt{3}K(1-\eta)$. Results are averaged over ten random configurations. Numbers beside the curves are the values of corresponding $\eta$. 

FIG. 7. A diagram showing the minimum and maximum energy states for the system.
IV. SYMMETRY ANALYSIS

In attempting to reduce the number of independent elastic constants describing the stretched network, we notice that in the unstretched network the elastic constant \( C_{ab\gamma r} \) is invariant under the index exchange \((a\beta \leftrightarrow \gamma \tau)\) and \((a\alpha \leftrightarrow \beta)\), \((\gamma \leftrightarrow \tau)\). Because of the initial stress there are now first-order terms in \( \varepsilon_{ab} \) so that these symmetry relations need to be modified. In Sec. II we derived all of the elastic constants for the pure triangular lattice in terms of \( L_0 / L \). Recalling Eqs. (13)–(20), we can find many relations between these quantities. We list three interesting ones below:

\[
T = C_{44} - \tilde{C}_{44}, \quad \text{i.e.,} \quad T = 2\mu_r, \quad (27)
\]

\[
C_{11} - C_{12} = C_{44} + \tilde{C}_{44}, \quad \text{i.e.,} \quad \mu_2 = b, \quad (28)
\]

\[
C_{12} = C_{44}. \quad (29)
\]

In the next three subsections we will discuss the validity of Eqs. (27)–(29) in terms of the underlying physics on the dilated stretched triangular network.

A. Rotational invariance of the strain energy

First we consider Eq. (27) which relates the tension to the second-order elastic constants \( C_{44} \) and \( \tilde{C}_{44} \). From the macroscopic invariance of the strain-energy density (4) under an infinitesimal rigid rotation, Huang has shown macroscopically that\(^{15}\)

\[
C_{ab\gamma r} - C_{b\alpha r} = S_{ab} \delta_{\alpha \gamma} - S_{ab} \delta_{\gamma r}. \quad (30)
\]

By substituting \( a\beta \gamma r \) with \( xyxy \) and \( yxyx \) in (30), we find

\[
\begin{align*}
C_{xyxy} - C_{yxyx} &= S_{yy}, &(31a) \\
C_{yxyx} - C_{xyxy} &= S_{xx}. &(31b)
\end{align*}
\]

These two equations are equivalent to each other since

\[
S_{xx} = S_{yy} = T
\]

and

\[
C_{xyxy} = C_{yxyx} = C_{44},
\]

\[
C_{yxyx} = C_{xyxy} = \tilde{C}_{44},
\]

so that both (31a) and (31b) lead to Eq. (27). Because the derivation of Huang is macroscopic, it should apply equally well to random and nonrandom systems.

We now reexamine this relation from the microscopic point of view. Consider an infinitesimal rigid rotation:

\[
u_i = \omega \times R_i + \frac{1}{2} \omega \times (\omega \times R_i), \quad (32)
\]

where it is important for our discussion to retain terms up to second order\(^{14,15}\) in \( \omega \) and not just the usual infinitesimal rotation. In our two-dimensional system, \( \omega \) is taken along the direction perpendicular to the network plane such that \( u_i \) can be written as

\[
u_i = \omega \times R_i - \frac{1}{2} \omega^2 R_i. \quad (33)
\]

Replacing \( u_i \) by (33) in the potential expansion (2), we get

\[
V = \frac{1}{2} \sum_{ij} K_{ij} (R_{ij} - L_0)^2 - \frac{\omega^2}{2} \sum_{ij} K_{ij} (R_{ij} - L_0) R_{ij} + \frac{1}{2} \sum_{ij} K_{ij} (1 - L_0 / R_{ij}) \omega^2 R_{ij}^2 + O(\omega^3). \quad (34)
\]

Here we have explicitly reinstated the \( ij \) dependence of the \( K_{ij} \) so that the result will be completely general for any spring constants. The second and third terms cancel under this rigid rotation so that \( V \) is invariant as expected. These two cancelling terms are just the tension \( T \) and second-order elastic constant \( C_{44} - \tilde{C}_{44} \). We divide the \( u_i \) corresponding to the rigid rotation into two parts:

\[
u_i^{(1)} = - \frac{\omega^2}{2} R_i = \epsilon R_i, \quad (35)
\]

where \( \epsilon = - \omega^2 / 2, \) and

\[
u_i^{(2)} = \phi \times R_i, \quad (36)
\]

where \( \phi = \omega \). In the computer simulation, we actually investigated the effect of \( u_i^{(1)} \) and \( u_i^{(2)} \) separately. For \( u_i^{(1)} \) the transformation is

\[
x' = x (1 - \epsilon), \quad (37)
\]

\[
y' = y (1 - \epsilon). \quad (37)
\]

The tension is extracted from the linear term in \( \epsilon \), i.e., the second term in Eq. (34), using (35):

\[
T = \sum_{ij} K_{ij} (R_{ij} - L_0) R_{ij} / 2 A. \quad (38)
\]

Note that the coefficient of the second-order term in \( \epsilon \) is the bulk modulus. On the other hand, \( u_i^{(2)} \) corresponds to a homogeneous infinitesimal rotation:

\[
x' = x + \phi y, \quad (39)
\]

\[
y' = y - \phi x. \quad (39)
\]

FIG. 7. Numerical simulation confirmation of the relation \( C_{44} - \tilde{C}_{44} = T \) for dilute networks with various \( \eta \) as shown. The dashed lines are for the guidance of the eye only.
from which we get the second-order elastic constant
\[ \mu_r = \sum_{(ij)} K_{ij} \left( 1 - L_0 / R_{ij} \right) R_{ij}^2 / 4 A, \]
(40)
where the corresponding linear-order term in \( \omega \) is always zero. Comparing Eqs. (38) and (40) we see that \( T = 2\mu_r \). Figure 7 shows a numerical confirmation of Eq. (27) from our numerical simulations on diluted networks. We have also confirmed that the tension is given correctly by Eq. (38) by comparing it with the tension found using Eq. (25). We have found the expression (38) for the tension to be particularly useful as it requires knowledge of only the equilibrium structure. This result can also be derived more conventionally by drawing an imaginary line through the sample and writing down the force per unit length perpendicular to this line produced by the bonds that cut the line. From this viewpoint it is to be expected that the tension is obtainable from the equilibrium structure, and that it is unnecessary to take a first derivative using equations like (24)–(26).

B. Isotropy for the sound velocities

Now let us consider Eq. (28). With the given energy of deformation (4), the wave equation is
\[ \rho \ddot{u}_\alpha (\mathbf{R}) = \sum \sum_{\beta \gamma} C_{\alpha \beta \gamma \rho} \frac{\partial u_\gamma (\mathbf{R})}{\partial R_\rho \partial R_\tau}. \]
(41)
For a plane-wave solution,
\[ u(\mathbf{R}) = u \exp(i \mathbf{k} \cdot \mathbf{R} - i \omega t), \]
(42)
we find

FIG. 8. Numerical simulation confirmation of the relation \( C_{44} = C_{11} - C_{12} \) for diluted networks with various \( \eta \) as shown. The dashed lines are for the guidance of the eye only.

FIG. 9. Numerical simulation result showing the invalidity of the possible Cauchy relation \( C_{12} = C_{44} \) for various \( \eta \) as shown.
\[ \rho \omega^2 u_a = \sum_y \left[ \sum_{\beta, \gamma} \mathbf{C}_{\alpha \beta \gamma} k_\beta k_\gamma u_\gamma \right]. \]

Specifically, for the triangular net under tension (43) becomes

\[ \rho \omega^2 \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} C_{11}k_x^2 + C_{44}k_y^2 & (C_{12} + \bar{C}_{44})k_x k_y \\ (C_{12} + \bar{C}_{44})k_x k_y & C_{44}k_x^2 + C_{11}k_y^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \]

In the (1,0) direction, the two eigenvalues are

\[ \rho \omega^2 / k^2 = C_{11}, C_{44}, \]

corresponding to longitudinal and transverse waves, respectively. In the (1,1) direction where \( k_x = k_y = k / \sqrt{2}, \) the two eigenvalues are

\[ \rho \omega^2 / k^2 = \frac{C_{11} + C_{44} \pm (C_{12} + \bar{C}_{44})}{2}. \]

The sound velocities for the triangular net are same in all directions, by equating (45) and (46), we have

\[ C_{11} + C_{44} + (C_{12} + \bar{C}_{44}) = 2C_{11}, \]

\[ C_{11} + C_{44} - (C_{12} + \bar{C}_{44}) = 2C_{44}. \]

Both of these equations are the same and are equivalent to Eq. (28). Note that the perfect triangular net under tension is elastically isotropic. The introduction of random defects that have no preferred direction does not alter this isotropy. In Fig. 8 we present a numerical check on the isotropy condition (28) for the diluted system.

### C. Cauchy relation

Finally, in this section, we study the validity of the Cauchy’s relation on the diluted networks. For networks with central forces only, and where every site is a center of inversion symmetry, Cauchy\(^{17}\) has shown that there are certain additional relations among the elastic constants. For high symmetry networks, like the triangular network, there is the single relation \( C_{12} = C_{44}. \) It has been shown to hold accuracy numerically\(^{4–6,18}\) but not analytically, that this relation also holds true when bonds are removed. We wonder if such a relation can hold more generally when the network is put under tension. In the absence of an external tension, \( C_{44} = \bar{C}_{44} \) and so the only possible generalization of the Cauchy relation would be that given by Eq. (29) which would be consistent with Eqs. (13)–(16). We decided to test this using our numerical results which is shown in Fig. 9. From these results, we can see that there is an obvious discrepancy between \( C_{12} \) and \( \bar{C}_{44} \) except for the limiting cases discussed previously. Thus we must abandon any hope of a generalized Cauchy relation under tension.

In summary, through the above analysis, we conclude that for the diluted stretched spring, Eqs. (27) and (28) are valid, but not the Cauchy relation (29). This reduces the six quantities \( E, T, B, b, \mu_s, \) and \( \mu \) to four independent quantities.

### V. Effective-Medium Theory

From work in the previous sections, we need four independent quantities \( E \) (static energy), \( T \) (external tension), \( B \) (bulk modulus), and \( \mu_s \) (shear modulus) to completely describe the elastic behavior of the depleted triangular network under tension. The tension is regarded as a calculated quantity because we find it convenient to fix the area per particle before dilution \( A = \sqrt{3}/2 \, L_0^2. \) The original (unstrained) area is \( A_0 = \sqrt{3}/2 \, L_0^2 \) and all quantities that have been determined are regarded as functions of \( L_0 / L \) and \( (A_0 / A) \). A Legendre transformation\(^{19}\) could be performed, if desired, so that everything is a function of the tension \( T \) rather than its conjugate distance \( L_0 / L \).

In Fig. 6 we show the energy \( E \) and the tension \( T \) as functions of \( p \) for various \( \eta \), where \( \eta = L_0 / L \). Similarly, Fig. 10 shows the bulk modulus \( B \) and the shear modulus \( \mu_s \) against \( p \) for several values of \( \eta \). All the data points are averaged over five samples. Note that for \( \eta = 0, B \) and \( \mu_s \) are exactly equal as shown in I. It is very clear that \( p^* \), where the elasticity vanishes, depends on \( \eta \). In Fig. 11 we show this phase boundary in the stretched region, i.e., \( 0 < \eta < 1 \), where two extreme cases \( \eta = 0 \) and \( \eta = 1 \) were known before. This boundary was obtained by determining where \( E \) and \( T \) go to zero. No elaborate procedures were used and the results for \( p^* \) have errors \( \pm 0.02 \). It can be seen that the phase boundary is very flat until about \( \eta = 0.05 \) when it begins to drop increasingly steeply until \( \eta = 1 \).

It can also be seen from Fig. 10 that \( B \) is almost a straight line for the whole range of \( p \) down to \( p^* \), while \( \mu_s \), as well as \( E \) and \( T \) which are shown in Fig. 6, have more curvature and tail near \( p^* \). This suggests that an effective-medium theory for the bulk modulus may be the best way to obtain an effective-medium expression for \( p^* \).

As a preliminary to developing an effective-medium theory, we introduce a single defect in the perfect triangular network, and let it relax. This determines the new equilibrium configuration and energy. From this equilibrium configuration, the other quantities of interest can be calculated as before. We took great care to get accurate numbers as with only one bond removed out of 1320 the changes are small. We checked that removing two bonds, that were not close together, doubled all the changes. It was important to average the results over the three possible bond directions the missing bond could have, in order to preserve the symmetry.

It is convenient to express the initial slope of the various elastic quantities by the value \( p \) where this extrapolated slope would cross the \( p \) axis.\(^{20}\) This gives a convenient dimensionless quantity which we designate by the symbols \( p_E, p_T, p_B, p_b, p_{\mu_s}, \) and \( p_{\mu_s} \), where the subscripts refer to the elastic quantity whose initial slope is being extrapolated. The results are plotted in Fig. 12. Again, there are two identical pairs, \( p_T \) and \( p_{\mu_s} \), \( p_B \) and \( p_{\mu_s} \), which arise from the rotational invariance and isotropy, respectively, discussed in Sec. V.

In the remaining part of this section, we will present a theoretical analysis of the single defect case that allows...
the initial slopes to be calculated approximately. This is in contrast to the case where the external tension is zero, when these initial slopes can be calculated exactly.\(^4\)\(^,\)\(^18\) The initial slope can also be calculated exactly when \(L_0 = 0.20\).

Let us consider a triangular network with one bond (for example, between sites 1 and 2) removed as sketched in Fig. 13. After relaxation, the distance between 1 and 2 changes from \(L\) to \(L_{eq}\). This change is appreciable because the external tension is in no sense a small quantity. This is in contrast to when the external tension is absent, when infinitesimal additional strains produce infinitesimal changes around the removed bond. This leads to the con-

**FIG. 10.** The bulk modulus \(B\) (square) and shear modulus \(\mu_s\) (diamond) are plotted against \(p\), after averaging over five samples. Here \(\mu_s\) is scaled by the factor of \(1/(1 - \eta/2)\) so that the results for both are the same at \(\eta = 1\).

**FIG. 11.** The phase boundary determined by the vanishing of the elastic constants (symbols). The solid line is the effective-medium-theory result \(p^* = p_B\).

**FIG. 12.** Computer simulation results for initial slopes, which are presented by the intercepts on the \(p\) axis extracted from the single defect case (see text). The subscripts of \(p_E, p_T, p_B, p_{\mu_s}, p_{\mu_T}, p_b\) stand for the corresponding quantity calculated.
can be calculated using standard methods when the separation between these sites is \( L \) so that the rest of the system is a perfect undistorted triangular network. This effective spring constant is then the same when the system with the removed bond is relaxed and the separation between 1 and 2 becomes \( L_{eq} \). This is a good approximation inasmuch as the potential shown in Fig. 14 is harmonic. The result for \( K_e \) is expressed in terms of \( a^* \) where we know that \( 0 < a^* < 1 \),

\[
K_e = \frac{K}{a^*} - K ,
\]

and \( a^* \) can be calculated exactly from the dynamical matrix of the perfect lattice,

\[
a^* = \frac{2K}{zN} \sum_k \text{Tr} \left[ \sum_{\delta} \left( 1 - \exp(iLk \cdot \delta) \right) \delta \delta^\dagger \cdot \mathcal{D}^{-1}(k) \right] ,
\]

where \( \mathcal{D}(k) \) is the Fourier transform of the dynamical matrix,

\[
\mathcal{D}(k) = \frac{\kappa}{\eta} \sum_{\delta} \left( 1 - \exp(iLk \cdot \delta) \right) \delta \delta^\dagger + K(1-\eta) \sum_{\delta} \left( 1 - \exp(iLk \cdot \delta) \right) I ,
\]

and \( I \) is the \( d \times d \) unit matrix. The detailed lattice integrals to calculate \( a^* \) on the triangular net are given in the Appendix.

If the distance between 1 and 2 is \( \bar{L} \), where \( \bar{L} \) is arbitrary, then the energy \( E \) of the complete system may now be written

\[
E = \frac{1}{2} K_e (L_{eq} - \bar{L})^2 + \frac{1}{2} K' (\bar{L} - L_0)^2 ,
\]

where \( K' \) is any additional spring that connects sites 1 and 2. The separation \( \bar{L} \) and energy \( E \) can be found by minimizing (52) with respect to \( \bar{L} \). For the special case when \( K' = K \), the perfect network is restored and so \( \bar{L} = L \) is the usual spacing between sites on the perfect triangular network. We find that

\[
L_{eq} = \frac{L - L_0 a^*}{1 - a^*} .
\]

Substituting back into the expression for the energy (52), we find the amount of energy that must be added to the one defect case to recover the no-defect or perfect lattice is

\[
\Delta E = \frac{1}{2} K_e \left[ \frac{L - L_0 a^*}{1 - a^*} - L \right] + \frac{1}{2} K (L - L_0)^2
\]

\[
= \frac{1}{2} k (L - L_0)^2 \left[ \frac{1}{1 - a^*} \right] ,
\]

which determines the initial slope of the energy,

\[
p_E = a^* .
\]

Now we can form an expression for the equilibrium energy \( E \) of the network with a small fraction \((1-p)\) of defects:
\[ E = \frac{1}{2} N z K L^2 (1 - \eta)^2 \left[ \frac{E}{1 - \eta^2} \right], \quad (56) \]

where the defects are far enough apart to be noninteracting. This is the main result of this section. From this result it is straightforward to calculate the initial slopes for the tension \( T \) and the bulk modulus \( B \) which are expressed as \( p_T \) and \( p_B \), respectively. For the initial slope for the tension, we use the fact that \( T = (\partial E)/(\partial L) \) so that we have

\[ T = T_0 [1 - (1 - p)/(1 - p_T)], \quad (57) \]

where

\[ 1/(1 - p_T) = \left[ 1 - \frac{\eta}{2} (1 - \eta) \frac{\partial a^*}{\partial \eta} \right] / (1 - a^*). \quad (58) \]

For the bulk modulus, we have

\[ AB = \frac{L^2}{4 A^2} \frac{\partial^2 E}{\partial L^2}, \quad (59) \]

where \( A \) is the area. We find

\[ B = B_0 [1 - (1 - p)/(1 - p_B)], \quad (60) \]

where

\[ 1/(1 - p_B) = \left[ 1 - \eta (1 - \eta^2) \frac{\partial a^*}{\partial \eta} \right] \]

\[ + \left[ \eta (1 - \eta) \frac{\partial a^*}{\partial \eta} \right]^2 \]

\[ + \frac{\eta^2}{2} (1 - \eta^2) \frac{\partial^2 a^*}{\partial \eta^2} \right] / (1 - a^*). \quad (61) \]

All of these results, \( p_E, p_T, \) and \( p_B \), as well as the corresponding simulation results, are plotted in Fig. 15. It can be seen that the agreement is excellent. This is further evidence that the harmonic approximation to the potential, which is the only approximation made, is good for all \( \eta \). Because the bulk modulus \( B \) is close to linear for all \( p > p^* \), we find that \( p^* = p_B \) where \( p_B \) is given by (61) to within the accuracy of our simulations. This is shown in Fig. 11. Clearly \( p^* \) cannot be exactly equal to \( p_B \) but it does provide a good approximation that should be useful in other lattices where simulations may not be available.

VI. CONCLUSIONS

We found that two kinds of percolation problem, conductivity percolation and rigidity percolation, can be combined together in the stretched-spring model. Our main conclusion is that the threshold for the new model changes continuously from \( p_{cen} \) to \( p_c \). The reason for this continuos evolution is related to a subtle but vital difference between this model and the Born model which changes discontinuously when the rotationally noninvariant forces are included. In our case we can write an effective Born model to describe the deviations from equilibrium. However, the parameters \( \alpha_{ij} \) and \( \beta_{ij} \) are not the same for all the uncoupled bonds. Rather,

\[ \alpha_{ij} = K, \]

\[ \beta_{ij} = K (1 - L_0 / L_{ij}), \]

so that the \( \beta \) force depends on the actual length \( L_{ij} \) of the \( ij \) bond and so varies throughout the lattice. This is in contrast to the perfect lattice [see Eq. (3)] and the previously studied dilute Born model. An effective-medium theory has been derived for this phase boundary. We have obtained a complete set of data from simulations for the four independent elastic quantities on the bond depleted triangular network. These are the energy, tension, bulk modulus, and shear modulus.

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APPENDIX: DETAILS OF EFFECTIVE-MEDIUM THEORY ON THE TRIANGULAR NET

The value of \( a^* \) can be calculated from the formula

\[ a^* = \frac{2K}{2N} \sum_k \text{Tr} \left[ \sum_\delta [1 - \exp(iL_k \cdot \delta) \delta \delta \cdot D^{-1}(k)] \right], \quad (A1) \]

where \( D(k) \) is the Fourier transform of the dynamical matrix.

FIG. 15. Effective-medium-theory results for \( p_E, p_T, \) and \( p_B \) are plotted against \( \eta \) (solid lines). The symbols are the corresponding simulation results.
\[ D(k) = K \eta \sum_\delta [1 - \exp(iLk \cdot \delta)] \hat{\delta} \hat{\delta} + K(1 - \eta) \sum_\delta [1 - \exp(iLk \cdot \delta)]L, \quad (A2) \]

and \( L \) is the \( d \times d \) unit matrix.

After some matrix algebra, we get the result

\[ a^* = \frac{2d}{z} \eta^{-1} - \frac{2}{z}(\eta^{-1} - 1)S, \quad (A3) \]

where \( S \) is the sum in \( k \) space,

\[ S = \frac{K}{N} \sum_k \left[ \sum_\delta [1 - \exp(iLk \cdot \delta)] \text{Tr}[D^{-1}(k)] \right]. \quad (A4) \]

For the triangular lattice, we have

\[ a^* = \frac{3}{2} \eta^{-1} - \frac{1}{2}(\eta^{-1} - 1)S \quad (A5) \]

and

\[ S = \frac{1}{N} \sum_k F \frac{D_{xx} + D_{yy}}{D_{xx}D_{yy} - D_{xy}D_{yx}}, \quad (A6) \]

where

\[ D_{xx} = (1 - \eta)[6 - 2 \cos(2x) - 4 \cos x \cos y] + \eta[3 - 2 \cos(2x) - \cos x \cos y], \quad (A7a) \]

and

\[ D_{yy} = (1 - \eta)[6 - 2 \cos(2x) - 4 \cos x \cos y] + \eta(3 - 3 \cos x \cos y), \quad (A7b) \]

\[ D_{xy} = D_{yx} = \sqrt{3} \eta \sin x \sin y, \quad (A7c) \]

and

\[ F = 6 - 2 \cos(2x) - 4 \cos x \cos y, \quad (A7d) \]

where

\[ x = \frac{1}{2} k_x L, \]

\[ y = \frac{\sqrt{3}}{2} k_y L. \]

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