Absence of Ordering in Certain Isotropic Systems*

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The method of Mermin and Wagner [Phys. Rev. Lett. 17, 1133 (1966)] is used to show that one- and two-dimensional spin systems interacting with a general isotropic interaction

$$H = \frac{1}{2} \sum_{ij} g_{ij}(\Theta)(\mathbf{S}_i \cdot \mathbf{S}_j),$$

where the exchange interactions $g_{ij}(\Theta)$ are of finite range, cannot order in the sense that

$$\langle O_i \rangle = 0$$

for all traceless operators $O_i$ defined at a single site $i$. Mermin and Wagner have proved the above for the case $n = 1$ with $O_i = \mathbf{S}_i$, i.e., for the Heisenberg Hamiltonian. The proof allows us to rule out the possibility that a small isotropic biquadratic exchange $(\mathbf{S}_i \cdot \mathbf{S}_j)^2$ could induce ferromagnetism or antiferromagnetism in a two-dimensional Heisenberg system.

**INTRODUCTION**

In a very elegant piece of work, Mermin and Wagner\(^1\) were able to show that the expectation value of the spin at any site $i$, $\langle \mathbf{S}_i \rangle$ is rigorously zero for any Heisenberg interaction of finite range between the spins in one and two dimensions. The brackets $\langle \cdots \rangle$ denote thermal average. It is clear from their proof that the isotropy of the Hamiltonian plays the essential role and so we have extended the result to systems with higher-degree isotropic spin interactions.

It has recently become clear that although one- and two-dimensional systems exist in nature\(^2\) that are very nearly isotropic, they all have a small amount of anisotropy and it is almost certain that the ideal isotropy visualized in this paper will never actually be achieved. It is probably true that all one-dimensional systems with finite range interactions do not exhibit long-range order, but a small amount of anisotropy can induce a spontaneous magnetization in two dimensions as has recently been demonstrated\(^3\) in the two dimensional antiferromagnet $\text{K}_2\text{NiF}_4$. The present result allows us to state conclusively that it is the breaking of the two-dimensional isotropy in $\text{K}_2\text{NiF}_4$ that gives rise to the phase transition and not the existence of small biquadratic exchange of the form $(\mathbf{S}_i \cdot \mathbf{S}_j)^2$.

**PROPERTIES OF SPIN SPHERICAL HARMONICS**

It is convenient to work with the spin equivalents of the spherical harmonics rather than the $(\mathbf{S}_i \cdot \mathbf{S}_j)^n$. These spin spherical harmonics $C_{l,m}(\mathbf{S})$ are conveniently defined by an unnormalized form by the generating function\(^4\)

$$\sum_{m} t^m C_{l,m}(\mathbf{S}) = (-tS^+ + 2S^z + S^-/t)^l.$$
Hamiltonian

\[ H = \frac{1}{2} \sum_{i \neq j} J_{ij}^{(0)} C_n(S_i \cdot S_j) - \hbar \sum_i \exp(iK \cdot R_i) C_{L,0}(S_i), \]

(5)

where a small symmetry breaking field \( h \) is applied, that will be allowed to go to zero at the end of the calculation. The wave vector \( K \) is introduced so that all possible kinds of spiral and antiferromagnetic order can be eliminated. The operators \( A \) and \( B \) are defined by

\[ A = C_{L,-1}(-k - K), \]
\[ B = S^+(k), \]

(6)

where the Fourier transforms are defined in an analogous way to those in Ref. 1. We use Eqs. (3) to evaluate the commutators appearing in the Bogoliubov inequality and find that

\[ \frac{1}{2} \langle C_{L,-1}(k+K)C_{L,-1}^*(\textbf{k} - \textbf{K}) \rangle 
+ \langle C_{L,-1}^*(\textbf{k} + \textbf{K})C_{L,-1}(\textbf{k} - \textbf{K}) \rangle 
\geq \kappa TL(L+1) \langle O_L \rangle^2 
\times \left[ \sum_{k,n} \left[ J^{(0)}(\textbf{k} - \textbf{K}) - J^{(0)}(\textbf{k}) \right] \right. 
\left. \left[ n(n+1) - m(m+1) \right] \langle C_{n,m}(\textbf{k}) C_{n,m}^*(-\textbf{k}) \rangle 
+ \hbar L(L+1) \langle O_L \rangle^{-1} \right], \]

(7)

where the order parameter \( O_L \) is given by

\[ O_L = 1/N \sum_i \exp(iK \cdot R_i) C_{L,0}(S_i) \]

(8)

and the Fourier transform of the interaction is

\[ J^{(0)}(\textbf{k}) = N^{-1} \sum_{i,j} \exp(i\textbf{k} \cdot (\textbf{R}_i - \textbf{R}_j)) J_{ij}^{(0)}. \]

We proceed as in Ref. 1 by summing both sides of (7) over \( \textbf{k} \), using the inequality \( \sum_i |A_i| \leq \sum_i |A_i| \) together with

\[ \sum_i \langle C_{L,-1}(S_i)C_{L,-1}^*(S_i) \rangle 
+ \langle C_{L,1}(S_i)C_{L,1}^*(S_i) \rangle 
\leq \sum_i \langle C_L(S_i^2) \rangle = NC_L \langle S_i^2 \rangle 
\]

and

\[ \sum_{n=1} \left[ n(n+1) - m(m+1) \right] = 4/3n(n+1/2)(n+1) \]

and derive the inequality

\[ \frac{1}{2} \langle C_L(S_i^2) \rangle > \kappa TL(L+1) \langle O_L \rangle^{1/2} \sum_k \left( \kappa k^2 + \hbar L(L+1) \langle O_L \rangle \right)^{-1}, \]

(9)

where

\[ \alpha = 1/N \sum_{i \neq j} \left| J^{(0)}(\textbf{R}_i - \textbf{R}_j) \right|^2 \langle C_L(S_i^2) \rangle 4/3n(n+1)(n+1). \]

The structure of this equation is identical to Eq. (12) in Mermin and Wagner,1 and so we can proceed in a similar fashion to show that for sufficiently small fields \( h \)

\[ \left| \langle O_L \rangle \right| \left( \text{const/}T^{1/2} \right) \left| h \right|^{1/12} \text{ in 1 dimension} \]
\[ \left| \langle O_L \rangle \right| \left( \text{const/}T^{1/2} \right) \left| h \right|^{1/12} \text{ in 2 dimensions}, \]

(10)

where we have taken the thermodynamic limit \( N \to \infty \) in (9) and replaced the summation by an integral. We see from (10) that as \( |h| \to 0 \) at any finite temperature \( \langle O_L \rangle \to 0 \) also.

**CONCLUSION**

The Hamiltonian (5) contains the most general isotropic spin–spin interaction. We have shown that \( \langle O_L \rangle = 0 \), where \( L \) is arbitrary and also contains an arbitrary \( K \) vector. It is therefore clear that \( \langle O_L \rangle = 0 \), where \( O_L \) is any single site operator with zero trace. Thus we can rule out any ordering in the usual sense in one and two dimensional isotropic systems, although an ordering of pairs as envisaged by Stanley and Kaplan4 remains a possibility.

* Work supported by the U.S. Atomic Energy Commission.
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